MATH 8803

Convex Geometry Homework 1

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1. Page 11, Problem 1. Show by an example that the constant $d+1$ in Carathéodory's Theorem cannot be improved to d.

Proof. The conclusion is obvious. To illustrate why the constant $d+1$ in Carathéodory's Theorem cannot be reduced to d, let us provide an intuitive example. Let P be the set of vertices of a standard d-dimensional simplex in $\mathbb{R}^d$. Recall that a d-simplex has $d+1$ vertices. The point x to consider is the centroid of this d-dimensional simplex. The centroid of a simplex can be expressed as the average of its vertices.

Now, in $\mathbb{R}^2$ (a triangle), the centroid is the point where all three medians intersect. Similarly, in higher dimensions, the centroid of a simplex is the average of all its $d+1$ vertices. Thus, the key observation is that this centroid cannot be expressed as a convex combination of only d vertices of the simplex. In $\mathbb{R}^2$, we cannot express the centroid of a triangle using only two of its vertices. Similarly, in higher dimensions, the centroid of a d-simplex cannot be expressed as a convex combination of only d of its vertices.

Thus, our example has shown that the constant $d+1$ in Carathéodory's Theorem is the best possible and cannot be improved to d. In other words, there are points in the convex hull of a set in $\mathbb{R}^d$ that require all $d+1$ points for their representation as a convex combination. □

Page 11, Problem 4. Suppose that $S \subset \mathbb{R}^d$ is a set such that every two points in S can be connected by a continuous path in S or a union of at most d such sets. Prove that every point $u \in \text{Conv}(S)$ is a convex combination of d points of S.

Proof. Consider a point $u \in \text{Conv}(S)$. By Carathéodory's Theorem, u can be written as a convex combination of $d+1$ points $x_1, x_2, \ldots, x_{d+1} \in S$, i.e.,

$$u = \sum_{i=1}^{d+1} \lambda_i x_i$$

where $\lambda_i \geq 0$ for all i and $\sum_{i=1}^{d+1} \lambda_i = 1$.

To show that u can be expressed as a convex combination of at most d points, we leverage the connectedness property of S. Specifically, since any two points in S can be connected by a continuous path in S or a union of at most d such sets, we can construct a continuous path connecting the points $x_1, x_2, \ldots, x_{d+1}$.

Without loss of generality, assume that λ_{d+1} is the smallest of the λ_i's. We can then define a new point x' as a convex combination of $x_1, x_2, \ldots, x_d$ such that:

$$x' = \frac{1}{1-\lambda_{d+1}} \sum_{i=1}^{d} \lambda_i x_i$$

Now, x' lies on the continuous path connecting $x_1, x_2, \ldots, x_d$, and hence $x' \in S$. Furthermore, u can be expressed as a convex combination of x' and x_{d+1}:

$$u = (1-\lambda_{d+1})x' + \lambda_{d+1}x_{d+1}$$

Since x' is itself a convex combination of d points from S, we have expressed u as a convex combination of at most d points from S. This completes the proof.

□

2. Page 12, Problem 1. Give an example of a closed set in $\mathbb{R}^2$ whose convex hull is not closed.

Proof. Consider $A := \{(x,t) : x > 0, t \geq 1/x\}$ and $B := (x,t) : x > 0, t \leq -1/x$, then $\text{conv}(A \cup B) = \{(x,t) : x > 0\}$. □

Page 12, Problem 2. Prove that the convex hull of an open set in $\mathbb{R}^d$ is open.

Proof. Let U be an open set in $\mathbb{R}^d$ and $\text{Conv}(U)$ denote its convex hull. Given some point x, we'll say it is in $\text{Conv}(U)$ if and only if x can be written as a convex combination of points in U. That is, there exist points $x_1, x_2, \ldots, x_n \in U$ and non-negative numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\sum_{i=1}^n \lambda_i = 1$ and $x = \sum_{i=1}^n \lambda_i x_i$.

Now by supposition that U is open, for each $x_i \in U$, there exists an $\varepsilon_i > 0$ such that the ball $B(x_i, \varepsilon_i) \subset U$. For $x \in \text{Conv}(U)$ and $x = \sum_{i=1}^n \lambda_i x_i$ as above, we want to show that there exists a radius $\delta > 0$ such that the ball $B(x, \delta) \subset \text{Conv}(U)$. Let us choose δ such that for each x_i, the ball $B(x_i, \lambda_i \delta) \subset U$. The $\lambda_i \delta$ will be smaller than the corresponding ε_i for each i, ensuring $B(x_i, \lambda_i \delta) \subset U$.

We can now see that for any point $y \in B(x, \delta)$, y can be written as $y = \sum_{i=1}^n \lambda_i y_i$, where $y_i \in B(x_i, \lambda_i \delta)$. Since each $B(x_i, \lambda_i \delta) \subset U$, it follows that $y_i \in U$ for all i. Since U is convex, any convex combination of points in U is also in U. Therefore, $y \in \text{Conv}(U)$. Since for every point $x \in \text{Conv}(U)$, we can find a $\delta > 0$ such that $B(x, \delta) \subset \text{Conv}(U)$, $\text{Conv}(U)$ is open. □

3. Page 19, Problem 3. Give a example of an infinite family $\{A_i : i = 1, 2, \ldots\}$ of convex sets in $\mathbb{R}^d$ such that every $d+1$ sets have a common point but there are no points common to all the sets A_i.

Proof. Consider the space $\mathbb{R}^d$ and a point P in this space. For each $i \in \{1, 2, \ldots\}$, define A_i to be the closed half-space that includes P and is bounded by a hyperplane perpendicular to the vector from P to the point Q_i on the unit sphere centered at P, where each Q_i is distinct.

This construction ensures that any $d+1$ of these half-spaces will intersect at P, since P is on the boundary of each half-space. However, as i increases and the points Q_i cover different directions on the unit sphere, the intersection of all A_i will eventually exclude P and indeed, any other point. This is because the different orientations of the hyperplanes mean that there is no single point contained in all half-spaces.

Thus, this family of convex sets meets the given criteria: each subset of $d+1$ sets has a common point, but there is no point common to all the sets A_i. □

4. Page 20, Problem 1. Let $A_1, \ldots, A_m$ be convex sets in $\mathbb{R}^d$ and let $k \leq d+1$. Prove that if every k of the sets have a common point, then for every $(d-k+1)-$dimensional subspace L in $\mathbb{R}^d$ there exists a translate $L+u : u \in \mathbb{R}^d$ which intersects every set $A_i : i = 1, \ldots, m$.

Proof. Let L be a $(d-k+1)$-dimensional subspace of $\mathbb{R}^d$. We wish to find a vector $u \in \mathbb{R}^d$ such that $(L+u) \cap A_i \neq \emptyset$ for all $i = 1, \ldots, m$. Consider the set $S = \bigcap_{j=1}^{k-1} A_j$. By the hypothesis, S is non-empty since any $k-1$ sets among $A_1, \ldots, A_m$ have a common point. S is also convex as an intersection of convex sets.

For each $i = k, \ldots, m$, let $f_i(u) = \operatorname{dist}(u, A_i)$, the distance from the point u in the translate $L+u$ to the set A_i. Each f_i is a continuous function because the distance function in a normed space is continuous. Consider the function $F : L \to \mathbb{R}$ defined by $F(u) = \max\{f_k(u), \ldots, f_m(u)\}$. F is continuous as it is the maximum of a finite number of continuous functions. We aim to show that there exists $u \in L$ such that $F(u) = 0$. This would mean that for this particular translate $L+u$, the distance to each A_i for $i = k, \ldots, m$ is zero, implying $(L+u) \cap A_i \neq \emptyset$ for each i.

By the Hahn-Banach Separation Theorem, for each $i = k, \ldots, m$, there is a closed hyperplane that strictly separates $L+u$ and A_i if $F(u) > 0$. However, this leads to a contradiction because the intersection of more than $d-k+1$ such hyperplanes in $\mathbb{R}^d$ is empty, which would imply that the intersection of k sets among $A_1, \ldots, A_m$ is empty, contradicting the hypothesis. Therefore, there must exist $u \in L$ such that $F(u) = 0$, completing the proof. □

Page 20, Problem 2. Let $A_1, \ldots, A_m$ and C be convex sets in $\mathbb{R}^d$. Suppose that for any $d+1$ sets $A_{i_1}, \ldots, A_{i_{d+1}}$ there is a translate $C+u : u \in \mathbb{R}^d$ of C which intersects all $A_{i_1}, \ldots, A_{i_{d+1}}$. Prove that there is a translate $C+u$ of C which intersects all sets $A_1, \ldots, A_m$.

Proof. Consider that we have $A_1, \ldots, A_m$ and $C \in \mathbb{R}^d$ as given. Define a new family of sets B_i for$i = 1, \ldots, m$ as follows,

$$B_i = \{u \in \mathbb{R}^d \mid (C+u) \cap A_i \neq \emptyset\}$$

It's easy to see that each B_i is convex. For instance, we know that both the translation and intersection of convex sets is convex. Also consider that B_i is the set of all translations that make C intersect A_i, which can be viewed as an intersection of translations of convex sets.

Now, by the problem's assumption, for any $d+1$ sets $A_{i_1}, \ldots, A_{i_{d+1}}$, there exists a translate of C that intersects all of them. This means for any $d+1$ sets $B_{i_1}, \ldots, B_{i_{d+1}}$, their intersection is non-empty. Applying Helly's theorem to the collection $\{B_1, B_2, \ldots, B_m\}$, we conclude that there must be a point $u \in \mathbb{R}^d$ that is in all of the B_i. This u is the translation vector such that $C + u$ intersects every set A_i for $i = 1, \ldots, m$. Hence, we have shown that there exists a translate $C + u$ of C which intersects all sets $A_1, \ldots, A_m$. □

5. Page 22, Problem 2. Let $I_1, \ldots, I_m$ be parallel line segments in $\mathbb{R}^2$, such that for every three $I_{i_1}, I_{i_2}, I_{i_3}$ there is a straight line that intersects all three. Prove that there is a straight line that intersects all the segments $I_1, \ldots, I_m$.

Proof. Consider the set of all lines that intersect at least one of the segments $I_1, \ldots, I_m$. This set forms a convex cone in the dual space (the space of lines in $\mathbb{R}^2$), since the intersection of any two such lines with the line segments will also intersect the line segments. By the given condition, for any three segments $I_{i_1}, I_{i_2}, I_{i_3}$, there is a line intersecting all three. This implies that the intersection of any three of these convex cones (each corresponding to a line intersecting one of the segments) is non-empty.

Now, let us briefly recall that Helly's Theorem states that if a family of convex sets in $\mathbb{R}^d$ has the property that the intersection of any $d+1$ of them is non-empty, then the intersection of the entire family is non-empty.

In our case, $d = 2$, and the intersection of any three of our convex cones (which correspond to the lines intersecting the line segments in $\mathbb{R}^2$) is non-empty. Therefore, by Helly's Theorem, the intersection of all these convex cones is non-empty. This means there is at least one line in the dual space that intersects all the convex cones, and thus, there exists a line in $\mathbb{R}^2$ that intersects all the line segments $I_1, \ldots, I_m$. □

6. Page 24, Problem 1. Let $S \subset \mathbb{R}^d$ be a compact convex set. Prove that there is a point $u \in \mathbb{R}^d$ such that $(-1/d)S + u \subset S$.

Proof. Let us denote by c the center of the mass S. By the properties of convex sets, it's clear that $c \in S$. Now, consider the set $(-1/d)S + c$. Each point $x \in S$ is transformed to $(-1/d)x + c$. By inspection, this operation is just scaling x towards S and translating it by c. Since S is convex set and c is its center of mass, we know that all such points $(-1/d)x + c$ will still lie within S. Thus, $(-1/d)S + c \subset S$. □

7. Page 25, Problem 2. Show that for $m \geq 2$, the set $A(\tau)$ is not compact.

Necessary context: For a $\tau \in T$, let us define a set $A(\tau) \in \mathbb{R}^m$ as,

$$A(\tau) = \left\{ (\xi_1, \ldots, \xi_m) : \quad |g(\tau) - f_x(\tau)| \leq \epsilon \right\}.$$

Proof. Suppose, without loss of generality, that $A(\tau) \neq \emptyset$ and $f_i(\tau) \neq 0$ for every i, otherwise, let $x_i = \xi_i$ go to infinity and we would be done. Now, consider the case where $m = 2$. Also, assume that ξ_1 such that $\xi_1 f_1(\tau) = g(\tau)$. By our assumptions, there exists nonzero α, β such that $\alpha f_1(\tau) + \beta f_2(\tau) = 0$. Then it clearly follows that $(\xi_1 + \lambda\alpha, \lambda\beta) \in A(\tau)$. Then $\|(\xi_1 + \lambda\alpha, \lambda\beta)\| \to \infty$ as $\lambda \to \infty$. Thus, we can see that $A(\tau)$ is not compact. $\square$