MATH 8803

Convex Geometry Homework 2

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1. Page 42, Problem 5 Let $A \subset V$ be an affine subspace of dimension n. Prove that the maximum number of affinely independent points in A is $n+1$.

Proof. Consider a set of points $\{p_0, p_1, \ldots, p_k\}$ in a vector space V is said to be affinely independent if the set of vectors $\{p_1 - p_0, p_2 - p_0, \ldots, p_k - p_0\}$ is linearly independent in V. This means that no point in the set can be expressed as an affine combination of the others, where an affine combination of points is a linear combination of the points where the coefficients sum to 1.

Given that A is an affine subspace of dimension n, it means that any maximal set of linearly independent vectors in $A - A$ (the set of differences of points in A) has n vectors. Now, consider the base case for $n = 0$, A is a single point, and the maximum number of affinely independent points is $1 = 0 + 1$, which holds trivially.

To prove the inductive step, assume we have a set of k affinely independent points in A, $\{p_0, p_1, \ldots, p_{k-1}\}$, where $k \leq n + 1$. The vectors $\{p_1 - p_0, \ldots, p_{k-1} - p_0\}$ are linearly independent by the definition of affine independence. If we attempt to add another point p_k to this set such that the set remains affinely independent, then $p_k - p_0$ must be linearly independent of the existing set of vectors $\{p_1 - p_0, \ldots, p_{k-1} - p_0\}$.

Since A is of dimension n, the maximal number of linearly independent vectors in $A - A$ is n. This implies that we cannot have more than n vectors that are linearly independent. Thus, the maximum number of affinely independent points is $n + 1$. □

2. Page 43, Problem 3 Prove that the projection $pr : V \to V/L$ is indeed a linear transformation, that its image is the whole space V/L and that its kernel is L.

Proof. To prove that pr is a linear transformation, we must show additivity and scalar multiplication. Indeed, for any $u, v \in V$, we have that

$$pr(u + v) = (u + v) + L = (u + L) + (v + L) = pr(u) + pr(v)$$

which clearly preserves addition. For any scalar c and any $v \in V$,

$$pr(cv) = cv + L = c(v + L) = c \cdot pr(v)$$

demonstrating that pr preserves scalar multiplication. Now, the image of pr consists of all equivalence classes $v + L$ for $v \in V$. Since every element V/L is an equivalence class of the form $v+L$, it follows that pr is the whole space V/L. Now, considering the kernel, the kernel of pr consists of all vectors $v \in V$ such that $pr(v) = 0 + L = L$, which implies $v + L = L$. So, $v \in L$ clearly since $v + L = L$ if and only if v is an element contained in L. Conversely, every element of L clearly maps to L under pr, showing that the kernel of pr is exactly L. □

3. Page 47, Problem 2 Let $V = \mathbb{R}_\infty$ be the vector space of all infinite sequences $x = (\xi_1, \xi_2, \ldots)$ of real numbers such that all but finitely many terms ξ_i are zero. One can think of $\mathbb{R}_\infty$ as the space of all univariate polynomials with real coefficients. Let $A \subset V \setminus \{0\}$ be the set of all sequences x where the last non-zero term is strictly positive. Prove that $0 \notin A$, that A is convex, that A is not algebraically open, and that there are no affine hyperplanes H such that $0 \in H$ and H isolates A.

Proof. To prove convexity, let us consider any two sequences $x = (\xi_1, \xi_2, \ldots)$ and $y = (\eta_1, \eta_2, \ldots)$ in A and any scalar $\lambda \in [0, 1]$. The sequence $z = \lambda x + (1 - \lambda)y$ is a linear combination of x and y. Since both x and y have their last non-zero term strictly positive, and since a linear combination with positive coefficients preserves the sign of the last non-zero term, z also has its last non-zero term strictly positive, implying $z \in A$. Hence, A is convex. Since a set is algebraically open if, for every point x in the set, there exists an $\epsilon > 0$ such that the ball $B(x, \epsilon) \subset A$. Consider any sequence $x \in A$ and any $\epsilon > 0$. There exists a sequence y not in A (for instance, by changing the sign of the last non-zero term of x to negative) such that the norm $\|x - y\| < \epsilon$. This implies that $B(x, \epsilon)$ cannot be entirely contained in A, proving that A is not algebraically open.

Finally, since an affine hyperplane H in V can be described as the set of points x satisfying $f(x) = c$ for some linear functional f and constant c. Since $0 \in H$, we have $f(0) = c$. However, for any linear functional f and any $x \in A$, there exists a scalar $\lambda > 0$ such that $\lambda x \in A$ and $f(\lambda x) = \lambda f(x) \neq c$ for sufficiently large or small λ, contradicting the assumption that H isolates A. Thus, there are no affine hyperplanes H with $0 \in H$ that isolate A.

□

4. Page 50, Problem 7 Prove that every non-empty compact convex set in $\mathbb{R}^d$ has an exposed point.

Proof. Let K be a non-empty compact convex set in $\mathbb{R}^d$. By the supporting hyperplane theorem, for any point $x \in \partial K$, the boundary of K, there exists at least one supporting hyperplane H such that $x \in H$ and K lies entirely on one side of H. Since K is compact, the extreme value theorem guarantees that every continuous function attains its maximum and minimum on K. Consider the function $f_x(y) = \langle y, x \rangle$ for a fixed $x \in \mathbb{R}^d$ and $y \in K$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^d$. The function $f_x(y)$ is continuous in y and thus attains its maximum and minimum on K. The points at which these extrema are attained are exposed points, as they are points where the supporting hyperplane, defined by the gradient of f_x at these points, touches K at a single point. Therefore, K must have at least one exposed point, completing the proof. □

5. Page 53, Problems 1,2 Prove that the set of extreme points of a closed convex set in $\mathbb{R}^2$ is closed. Furthermore, construct an example of a compact convex set $K \subset \mathbb{R}^3$ such that $\text{ex}(K)$ is not closed.

Proof. Let C be a closed convex set in $\mathbb{R}^2$, and let E denote its set of extreme points. Suppose E is not closed. Then there exists a sequence $\{x_n\}$ of points in E converging to a point $x \in C$ such that $x \notin E$. Since x is not an extreme point, it can be written as a convex combination of two distinct points in C, say $x = \lambda y + (1-\lambda)z$ for some $y, z \in C$, $y \neq z$, and $0 < \lambda < 1$. However, this contradicts the assumption that each x_n is an extreme point, as extreme points cannot be expressed as a convex combination of other points in C. Hence, E must be closed.

For an example, consider the set

$$\{(x,y,z) : (z-1)^2+y^2 \leq (\frac{1-x}{2})^2, 1 \geq x \geq 0\} \cup \{(x,y,z) : (z-1)^2+y^2 \leq (\frac{1+x}{2})^2, -1 \leq x \leq 0\}.$$

Then since $(0,0,0)$ is a limit of the points on $\{(x,y,z) : x = 0, (z-1)2 + y_2 \leq 1\}$, all points in the set except for $(0,0,0)$ are extreme points, but $(0,0,0)$ can be written as a convex combination of $(-1,0,0)$ and $(0,0,1)$. □

6. Page 55, Problem 3 Prove that polytopes have finitely many faces.

Proof. Consider that a face F is a hyperplane H which isolates P and $F = H \cap P$. For every hyperplane H, it isolates P if every $x \in P$ satisfies $\langle c, x \rangle \leq \alpha$, and $F = H \cap P/ = \emptyset$ if there exists $x \in P$ with $\langle c, x \rangle = \alpha$. That is, H is one of the inequalities defining P, and by definition, there are finitely many of them. Thus it follows that there are finitely many faces.

□

7. Page 58, Problem 3 Prove that the set $F = \{X \in B_n : \xi_{11} = 0\}$ is a face of B_n of dimension $(n-1)^2 - 1$ and that $G = \{X \in B_n : \xi_{11} = 1\}$ is a face of B_n of dimension $(n-2)^2$.

Proof. Recall that F is of dimension $(n-1)^2 - 1$ by the fact that B_n has dimension $(n-1)^2$ and one entry being fixed as 1. Also, for G, B_n has dimension $(n-1)^2$ and so we can treat it as knowing the $(n-1)^2$ principal matrix the last row and column are known. While $\xi_{11} = 1$, by the definition of $B_n, \xi_{1i} = 0 = \xi_{i1}$ for every $i = 2, \ldots, n$, so the first row and column are also known that is, the dimension of G is $(n-2)^2$.

So suppose that F is convex since it is intersection of convex sets B_n and $\{X : X_{11} = 0\}$.Let $A \in F$, and consider $X, Y \in B_n, \lambda \in (0,1)$ such that $A = \lambda X + (1-\lambda)Y$.

Then by definition of B_n, $X_{11} \geq 0, Y_{11} \geq 0$. Since $\lambda X_{11} + (1-\lambda)Y_{11} = 0$, we know $X_{11} = 0 = Y_{11}$,so $X, Y \in F$, then F is a face of B_n by definition. For G we can see that G is convex for the same reason above. Let $A \in G$, and consider $X, Y \in B_n$, $\lambda \in (0,1)$

such that $A = \lambda X + (1-\lambda)Y$. By the definition of B_n, we know $X_{11} \leq 1$, $Y_{11} \leq 1$. Since $\lambda X_{11} + (1-\lambda)Y_{11} = 1$,we know $X_{11} = 1 = Y_{11}$,so $X, Y \in G$, then G is a face of B_n by definition. $\square$