MATH 8803

Convex Geometry Homework 3

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1. Page 60, Problem 4. Suppose that not all the coordinates of a are equal. Prove that $\dim P(a) = n - 1$.

Proof. Recall that the permutation polytope $P(a)$ lies in the affine hyperplane

$$H = \{(\xi_1, \dots, \xi_n) \in \mathbb{R}^n \,|\, \xi_1 + \cdots + \xi_n = \alpha_1 + \cdots + \alpha_n\}.$$

This hyperplane is defined by a single linear equation, indicating that H is an $n-1$ dimensional subspace of $\mathbb{R}^n$. To prove $\dim P(a) = n - 1$, we show that there are n affinely independent points within $P(a)$, implying its dimensionality is $n-1$ (since the dimension is one less than the number of affinely independent points).

Since not all coordinates of a are equal, permuting the coordinates of a yields vectors that are distinct. These vectors, including a itself, are points in $P(a)$ and lie within the hyperplane H. Consider any set of n such permutations, including a. This set forms a basis for H because no point can be written as an affine combination of the others, due to the distinctness of the coordinates in each permutation. Hence, we have identified n affinely independent points within $P(a)$, which lies in H. Therefore, the affine dimension of $P(a)$ is $n-1$, proving that $\dim P(a) = n - 1$. □

2. Page 67, Problem 1. Let $K \subset \mathbb{R}^d$ be a cone with a compact base. Prove that 0 is a face of K.

Proof. Recall that a cone K in $\mathbb{R}^d$ with a compact base can be represented as the set of all linear combinations of the form λx, where x belongs to the base B of the cone and $\lambda \geq 0$. The compactness of B ensures that K is closed and convex. To show that 0 is a face of K, we consider the definition of a face. A face of a convex set C is a convex subset F of C such that every closed line segment in C with an interior point in F has both endpoints in F.

The point 0 satisfies this definition for the cone K, as follows: For any closed line segment in K that contains 0 as an interior point, the line segment must be the trivial segment $[0, 0]$ since K, being a cone, emanates from 0 and contains no line segments that pass through 0 and extend in both directions.

Furthermore, 0 can be seen as the intersection of K with a supporting hyperplane that contains 0 and is orthogonal to any line passing through points of K. Such a hyperplane supports K at 0, making 0 a face of K. Therefore, we conclude that 0 is indeed a face of the cone K with a compact base. □

3. Page 67, Problem 2. Construct an example of a compact set $A \subset \mathbb{R}^2$ such that $co(A)$ is not closed.

Proof. Suppose we take $A = \{(x, \sin(1/x)) : x \in (0,1]\} \cup \{(0,y) : y \in [-1,1]\}$. This set is compact in $\mathbb{R}^2$, but its conic hull $co(A)$ is not closed. For instance, we can see that A is both bounded and closed. The graph of $y = \sin(1/x)$ for $x \in (0,1]$ is bounded, and the closure of this graph as x approaches 0 includes the line segment along the y-axis from -1 to 1, making A closed. Hence, A is compact by the Heine-Borel theorem.

The conic hull of A, $co(A)$, fails to be closed as it does not contain the limit point $(0,0)$, which can be approached by a sequence of points in $co(A)$ but cannot itself be represented as a conic combination of points in A. Thus, we've found an example set where its conic hull is not closed. □

4. Page 68, Problem 1. Prove that each hyperplane $H \subset \mathbb{R}^{d+1}$ such that $0 \in H$ intersects the moment curve $g(\tau)$ in at most d points.

Proof. A hyperplane H in $\mathbb{R}^{d+1}$ containing the origin can be defined by a linear equation of the form $a_1x_1 + a_2x_2 + \cdots + a_{d+1}x_{d+1} = 0$, where $(a_1, a_2, \ldots, a_{d+1}) \neq (0, 0, \ldots, 0)$ is a normal vector to the hyperplane. Recall that the moment curve $g(\tau)$ is given by $(\tau, \tau^2, \ldots, \tau^d, \tau^{d+1})$. For an intersection point between H and $g(\tau)$, we substitute $g(\tau)$ into the equation of H, yielding

$$a_1\tau + a_2\tau^2 + \cdots + a_d\tau^d + a_{d+1}\tau^{d+1} = 0.$$

This equation is a polynomial equation of degree $d+1$ in τ. By the Fundamental Theorem of Algebra, a polynomial of degree n has at most n roots, unless the polynomial is the zero polynomial. In our case, since not all a_i are zero, this is not the zero polynomial, and thus the equation has at most $d+1$ roots.

However, since the polynomial is of degree $d+1$, and we are considering the case $0 \in H$, which corresponds to one of the roots being trivially satisfied by $\tau = 0$, we are left with a polynomial of degree d that can have at most d non-zero roots. These roots correspond to the intersection points of H and $g(\tau)$, implying that there are at most d such intersection points. □

5. Page 68, Problem 3. Let $S^1 = \left\{(\cos\tau, \sin\tau) : 0 \leq \tau \leq 2\pi\right\}$ be the circle. Suppose that $d = 2k$ is even and let $h : S^1 \to \mathbb{R}^d$ be the closed curve

$$h(\tau) = (\cos\tau, \sin\tau, \cos 2\tau, \sin 2\tau, \ldots, \cos k\tau, \sin k\tau)\,,\ 0 \leq \tau \leq 2\pi.$$

Prove that each affine hyperplane $H \subset \mathbb{R}^d$ intersects the curve $h(\tau)$ in at most d points.

Proof. An affine hyperplane in $\mathbb{R}^d$ can be defined by the equation $a \cdot x = b$, where $a \in \mathbb{R}^d$ is a normal vector, x is a point in $\mathbb{R}^d$, and b is just a constant. The intersection of this hyperplane with the curve $h(\tau)$ requires solving

$$a_1 \cos\tau + a_2 \sin\tau + \ldots + a_{2k-1} \cos k\tau + a_{2k} \sin k\tau = b.$$

This equation is a trigonometric polynomial of degree k, implying at most k distinct roots, considering the periodicity of the trigonometric functions. Thus, the curve $h(\tau)$ intersects any affine hyperplane in $\mathbb{R}^d$ at most $d = 2k$ points.

$\square$

6. Page 71, Problem 1. Prove that one cannot find m points $\tau_1^*, \dots \tau_m^*$ in the interval $[0, 1]$ and m real numbers $\lambda_1, \dots, \lambda_m$ such that

$$\int_0^1 f(\tau) d\tau = \sum_{i=1}^m \lambda_i f(\tau_i^*)$$

for all polynomials f of degree $2m$.

Proof. Assume, by way of contradiction, that there exist points $\tau_1^*, \dots, \tau_m^*$ in $[0, 1]$ and real numbers $\lambda_1, \dots, \lambda_m$ such that

$$\int_0^1 f(\tau) d\tau = \sum_{i=1}^m \lambda_i f(\tau_i^*)$$

for all polynomials f of degree $2m$. Then let us consider $q(\tau) = (\tau - \tau_1^*)^2 (\tau - \tau_2^*)^2 \cdots (\tau - \tau_m^*)^2$, a polynomial of degree $2m$ that is zero at each τ_i^* and positive elsewhere in $[0, 1]$. According to our assumption, we should have

$$\int_0^1 q(\tau) d\tau = \sum_{i=1}^m \lambda_i q(\tau_i^*) = 0,$$

which contradicts the fact that $q(\tau)$, being strictly positive on $(0, 1)$ except at the τ_i^* points, has a strictly positive integral over $[0, 1]$. Therefore, no such points and coefficients can be found that satisfy the initial condition for all polynomials of degree $2m$, establishing the proof by contradiction.

$\square$

7. Page 71, Problem 3. A function

$$f(\tau_= \gamma_0 + \sum_{k=1}^d (\alpha_k \sin k\tau + \beta_k \cos k\tau) \ 0 \leq \tau \leq 2\pi$$

is called a trigonometric polynomial of degree at most d. Let ρ be a nonnegative continuous function on $[0, 2\pi]$ such that $\rho(0) = \rho(2\pi)$. Prove that there exists $d + 1$ points $0 \leq \tau_0^* < \dots < \tau_d^* < 2\pi$ and $d + 1$ nonnegative numbers $\lambda_0, \dots, \lambda_d$ such that the formula

$$\int_0^{2\pi} f(\tau) \rho(\tau) d\tau = \sum_{i=0}^d \lambda_i f(\tau_i^*)$$

is exact for any trigonometric polynomial of degree at most d.

Proof. Let us consider a trigonometric polynomial f of degree at most d, given by

$$f(\tau) = \gamma_0 + \sum_{k=1}^{d} (\alpha_k \sin k\tau + \beta_k \cos k\tau).$$

The function $\rho(\tau)$ is continuous and nonnegative on $[0, 2\pi]$ and satisfies the condition of $\rho(0) = \rho(2\pi)$, thus allowing for it to be a weight function for a weighted inner product space. Given $\rho(\tau)$, we can define an inner product on the space of trionometric polynomials of degree at most d as

$$\langle f, g \rangle = \int_0^{2\pi} f(\tau) g(\tau) \rho(\tau) d\tau.$$

Using the Gram-Schmidt process with this inner product, we can construct an orthogonal basis $\{p_0, p_1, \ldots, p_d\}$ for the space of trigonometric polynomials of degree at most d, where each p_i is a trigonometric polynomial of degree i. The zeros of $p_{d+1}(\tau)$, the first polynomial orthogonal to the space of degree at most d, allows for us to identify $d+1$ distinct products $\tau_0^*, \ldots, \tau_d^*$ in $[0, 2\pi)$. Finally, the weights λ_i can be determined by solving the linear system formed by enforcing the quadrature formula to be exact for the basis polynomials $p_i(\tau)$. That is, for each $i = 0, \ldots, d$,

$$\int_0^{2\pi} p_i(\tau) \rho(\tau) d\tau = \sum_{j=0}^{d} \lambda_j p_i(\tau_j^*).$$

This system is solvable because the matrix formed by evaluating p_i at τ_j^* is a Vandermonde matrix and is nonsingular, given that all τ_j^* are distinct. Each λ_i is precisely an integral of a nonnegative function over domain $[0, 2\pi]$ implying again nonnegativity. □