

# Lectures on 0/1-Polytopes

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## Abstract

These lectures on the combinatorics and geometry of 0/1-polytopes are meant as an *introduction* and *invitation*. Rather than heading for an extensive survey on 0/1-polytopes I present some interesting aspects of these objects; all of them are related to some quite recent work and progress.

0/1-polytopes have a very simple definition and explicit descriptions; we can enumerate and analyze small examples explicitly in the computer (e. g. using `polymake`). However, any intuition that is derived from the analysis of examples in “low dimensions” will miss the true complexity of 0/1-polytopes. Thus, in the following we will study several aspects of the complexity of higher-dimensional 0/1-polytopes: the doubly-exponential number of combinatorial types, the number of facets which can be huge, and the coefficients of defining inequalities which sometimes turn out to be extremely large. Some of the effects and results will be backed by proofs in the course of these lectures; we will also be able to verify some of them on explicit examples, which are accessible as a `polymake` database.

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## Introduction

These lectures are trying to get you interested in 0/1-polytopes. But I must warn you: they are mostly “bad news lectures” — with two types of bad news:

1. General 0/1-polytopes are complicated objects, and some of them have various kinds of extremely bad properties such as “huge coefficients” and “many facets,” which are bad news also with respect to applications.
2. Even worse, there are bad gaps in our understanding of 0/1-polytopes. Very basic problems and questions are open, some of them embarassingly easy to state, but hard to answer. So, 0/1-polytopes are interesting and remain *challenging*.

A good grasp on the structure of 0/1-polytopes is important for the “polyhedral combinatorics” approach of combinatorial optimization. This has motivated an extremely thorough study of some special classes of 0/1-polytopes such as the traveling salesman polytopes (see Grötschel & Padberg [31] and Applegate, Bixby, Cook & Chvátal [5]) and the cut polytopes (see Deza & Laurent [19], and Section 4). In such studies the question about properties of general 0/1-polytopes, and for complexity estimates about them, arises quite frequently and naturally. Thus Grötschel & Padberg [31] looked for upper bounds on the number of facets, and we can now considerably improve the estimates they had then (Section 2). One also asks for the sizes of the integers that appear as facet coefficients — and the fact that these coefficients may be huge (Section 5) is bad news since it means that there is a great danger of numerical instability or arithmetic overflow.

Surprisingly, however, properties of *general* 0/1-polytopes have not yet been a focus of research. I think they should be, and these lecture notes (expanded from my DMV-Seminar lectures in Oberwolfach, November 1997) are meant to provide support for this.

Of course, the distinction between “special” and “general” 0/1-polytopes is somewhat artificial. For example, Billera & Sarangarajan [9] have proved the surprising fact that *every* 0/1-polytope appears as a face of a TSP-polytope. Nevertheless, a study of the broad class of general 0/1-polytopes provides new points of view. Here it appears natural to look at *extremal* polytopes (e. g. polytopes with “many facets”), and at *random polytopes* and their properties.

Where is the difficulty in this study? The definition of 0/1-polytopes is very simple, examples are easy to come by, and they can be analyzed completely. But this simplicity is misleading: there are various effects that appear only in rather high dimensions ( $d \gg 3$ , whatever that means). Part of this we will trace to one basic linear algebra concept: determinants of 0/1-matrices, which show their typical behaviour — large values, and a low probability to vanish — only when the dimension gets quite large. Thus one rule of thumb will be justified again and again:

Low-dimensional intuition does not work!

Despite this (and to demonstrate this), our discussion in various lectures will take the low-dimensional situation as a starting point, and as a point of reference. (For example, the first lecture will start with a list of 3-dimensional 0/1-polytopes, which will turn out to be deceptively simple.)

However, examples are nevertheless important. The `polymake` project [28, 29] provides a framework and many fundamental tools for their detailed analysis. Thus, these lecture notes come with a library of interesting examples, provided as a separate section of the `polymake` database at

<http://www.math.tu-berlin.de/diskregeom/polymake/>

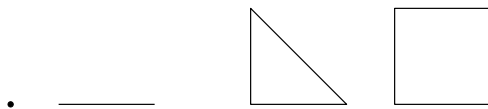
We will refer to examples in this database throughout. The names of the polytope data files are of the form `NN:d-n.poly`, where `NN` is an identifier of the polytope (e. g. initials of whoever supplied the example), `d` is the dimension of the polytope, `n` is its number of vertices. I invite you to play with these examples. (Also, I am happy to accept further contributions to extend this bestiary of interesting 0/1-polytopes!)

## 1 Classification of Combinatorial Types

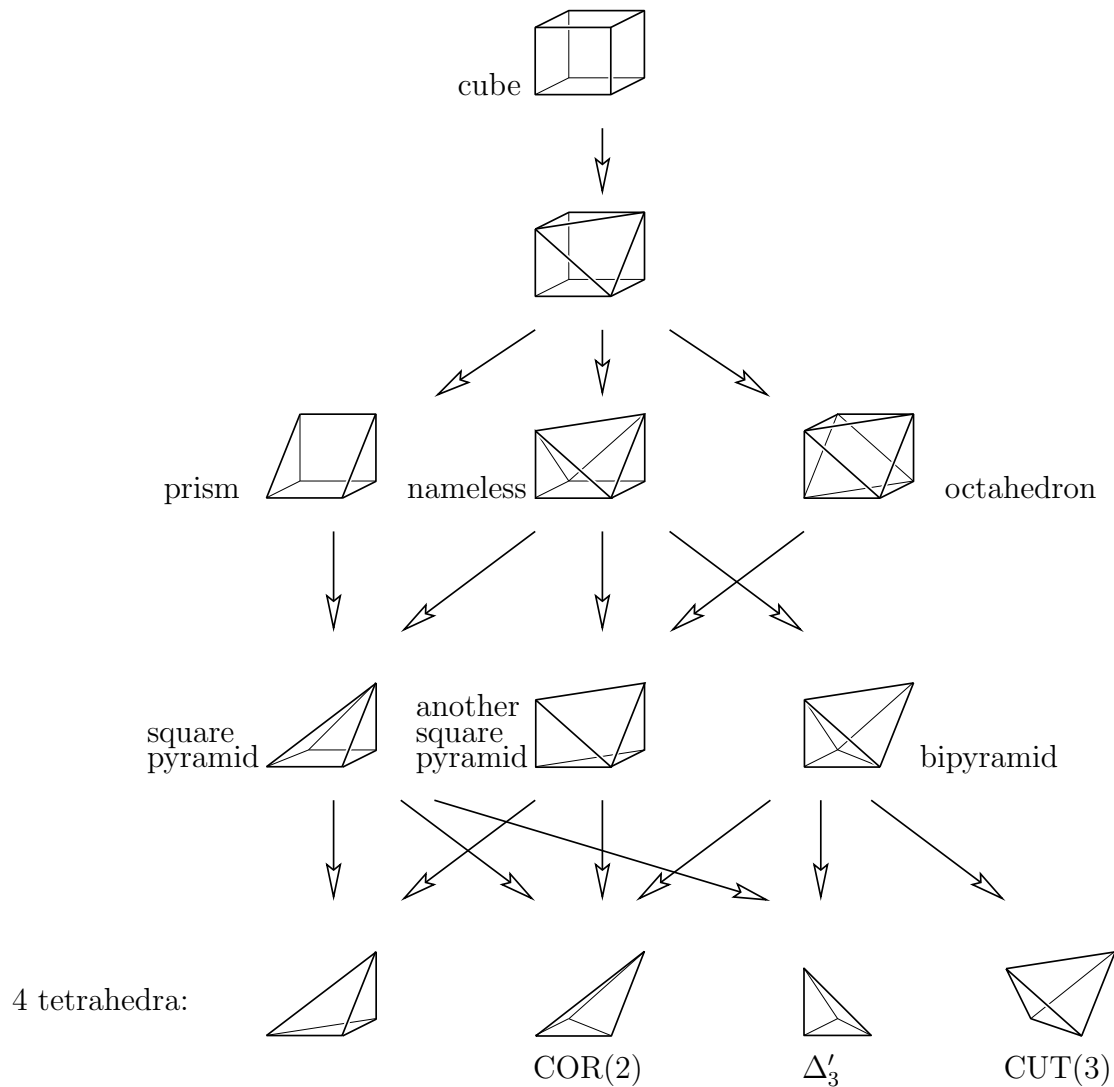
### 1.1 Low-dimensional 0/1-polytopes

0/1-polytopes may be defined as the convex hulls of finite sets of 0/1-vectors, that is, as the convex hulls of subsets of the vertices of the regular cube  $C_d = \{0, 1\}^d$ . Until further notice let's assume that we only consider full-dimensional 0/1-polytopes, so we have  $P = P(V) = \text{conv}(V)$  for some  $V \subseteq \{0, 1\}^d$ , where we assume that  $P$  has dimension  $d$ . We call two polytopes *0/1-equivalent* if one can be transformed into the other by a symmetry of the 0/1-cube.

Now 0/1-polytopes of dimensions  $d \leq 2$  are not interesting: we get a point, the interval  $[0, 1]$ , a triangle, and a square.



The figure below represents the classification of 3-dimensional 0/1-polytopes  $P \subseteq \mathbb{R}^3$  according to 0/1-equivalence. An arrow  $P \twoheadrightarrow P'$  between two of them denotes that  $P'$  is 0/1-equivalent to a *subpolytope* of  $P$ , that is,  $P' \sim P(V')$  and  $P = P(V)$  for some subset  $V' \subseteq V$ .



The full-dimensional 0/1-polytopes of dimension  $d = 4$  were first enumerated by Alexx Below: There are 349 different 0/1-equivalence classes.

In dimension 5 there are exactly 1226525 different 0/1-equivalence classes of 5-dimensional 0/1-polytopes. This classification was done by Oswin Aichholzer [2]: a considerable achievement, which was possible only by systematic use of all the symmetry that is inherent in the problem.

In October 1998, Aichholzer completed also an enumeration and classification of the 6-dimensional 0/1-polytopes up to 12 vertices. The complete classification of all 6-dimensional 0/1-polytopes is not within reach: in fact, even the output, a non-redundant list of all combinatorial types would be so huge that it is impossible to store or search efficiently: and thus it would probably<sup>1</sup> be useless.

<sup>1</sup>“Where a calculator like the ENIAC today is equipped with 18,000 vacuum tubes and weighs 30 tons, computers in the future may have only 1,000 vacuum tubes and perhaps weigh only  $1\frac{1}{2}$  tons.” — *Popular Mechanics*, March 1949, p. 258.

## 1.2 Combinatorial types

Many fundamental concepts of general polytope theory can be specialized to the situation of 0/1-polytopes. The following reviews the basic definitions and concepts. See for example [58, Lect. 0-3] for more detailed explanations.

0/1-polytopes, just as all other polytopes, can be described both in terms of their vertices (“ $\mathcal{V}$ -presentation”) and in terms of equations and facet-defining inequalities (“ $\mathcal{H}$ -presentation”). However, for 0/1-polytopes the first point of view yields the name, it gives the natural definition, and thus it also determines our starting point.

### Definition 1 (0/1-polytopes)

A 0/1-polytope is a set  $P \subseteq \mathbb{R}^d$  of the form

$$P = P(V) := \text{conv}(V) = \{V\mathbf{x} : \mathbf{x} \geq \mathbf{0}, \mathbf{1}^t\mathbf{x} = 1\}$$

where  $V \in \{0, 1\}^{d \times n}$  is a 0/1-matrix whose set of columns, a subset of the vertex set of the unit cube  $C_d = [0, 1]^d$ , is the *vertex set* of  $P$ .

### Notation 2

Here and in the following, we will extensively rely on vector and matrix notation. Our basic objects are column vectors such as  $\mathbf{x}, \mathbf{y}, \dots$ . Their transposed vectors  $\mathbf{x}^t, \mathbf{y}^t$  are thus row vectors. We use  $\mathbf{1}$  to denote a column vector of all 1s (whose length is defined by the context),  $\mathbf{0}$  to denote the corresponding zero vector, while  $\mathbf{e}_i$  denotes the  $i$ -th unit vector (of unspecified length). The product  $\mathbf{x}^t\mathbf{y}$  of a row with a column vector yields the standard scalar product, while  $\mathbf{x}\mathbf{y}^t$  is a product of a column vector with a row vector (of the same length), and thus represents a matrix of rank 1. Thus  $\mathbf{1}^t\mathbf{1} = n$  if  $\mathbf{1}$  has length  $n$ , while  $\mathbf{1}\mathbf{1}^t$  is a square all-1s matrix. Matrices such as  $V$  and their sets of columns are used interchangeably. A unit matrix of size  $n \times n$  will be denoted  $I_n$ .

It is hard to “see” what a 0/1-polytope looks like from looking at the matrix  $V$ . We have more of a chance to “understand” an example by feeding it to a computer and asking for an analysis. More specifically, we may present  $P(V)$  to the `polymake` system of Gawrilow & Joswig [28, 29] in terms of a file that contains the key word `POINTS` in its first line, and then the matrix  $(\mathbf{1}, V^t)$  in the following lines — the rows of this matrix give homogeneous coordinates for the vertices of  $P(V)$ .

### Example 3

For  $n \geq 1$ ,

$$\begin{aligned} \Delta_{n-1} &:= P(I_n) = \text{conv}(\{\mathbf{e}_1, \dots, \mathbf{e}_n\}) \\ &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}, \mathbf{1}^t\mathbf{x} = 1\} \subseteq \mathbb{R}^n \end{aligned}$$

is the *standard* simplex of dimension  $n - 1$ .

This is a *regular* simplex, since all its edges have the same length  $\sqrt{2}$ , but it is not full-dimensional, since it lies in the hyperplane given by  $\mathbf{1}^t\mathbf{x} = 1$ . Alternatively, we could

consider the simplex

$$\begin{aligned}\Delta'_n &= P(\mathbf{0}, I_n) = \text{conv}(\{\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_n\}) \\ &= \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}, \mathbf{1}^t \mathbf{x} \leq 1\} \subseteq \mathbb{R}^n,\end{aligned}$$

which is full-dimensional, but not regular for  $n \geq 2$ . In fact, in many dimensions (starting at  $n = 2$ ) there is no full-dimensional, regular 0/1-simplex at all. (See Problem 18.)

**Example/Exercise 4**

For  $V \in \{0, 1\}^{d \times n}$ , let  $\tilde{V} = \begin{pmatrix} V \\ I_n \end{pmatrix} \in \{0, 1\}^{(d+n) \times n}$ . Then

$$P(\tilde{V}) = \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in [0, 1]^{d+n} : \mathbf{y} \geq 0, \mathbf{1}^t \mathbf{y} = 1, \right. \\ \left. x_i = \sum_{j=1}^n v_{ij} y_j \text{ for } 1 \leq i \leq d \right\}$$

is an affine image of the  $(n-1)$ -dimensional standard simplex  $\Delta_{n-1}$ . (Prove this!) Thus for the 0/1-polytope  $P(\tilde{V}) \subseteq \mathbb{R}^{d+n}$  we have a complete description in terms of linear equations and inequalities. From this we get  $P(V)$  as the image of the projection

$$\begin{aligned}\pi : \mathbb{R}^{d+n} &\longrightarrow \mathbb{R}^d \\ \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} &\longmapsto \mathbf{x}\end{aligned}$$

that deletes the last  $n$  coordinates. Equivalently, to get  $P(V) = \pi(P(\tilde{V}))$  from  $P(\tilde{V})$  we must apply the operation “delete the last coordinate”  $n$  times.

**Theorem 5 ( $\mathcal{H}$ -presentations)**

Every 0/1-polytope  $P(V) \subseteq \mathbb{R}^d$  can be written as the set of solutions of a system of linear inequalities, that is, as

$$P(V) = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\}$$

for some  $n \in \mathbb{N}$ , a matrix  $A \in \mathbb{Z}^{n \times d}$ , and a vector  $\mathbf{b} \in \mathbb{Z}^n$ .

**Proof.** First, we need not deal with equations in the system that describes  $P(\tilde{V})$ , since these can be rewritten in terms of inequalities: the equation  $\mathbf{a}^t \mathbf{x} = \beta$  is equivalent to the two inequalities  $\mathbf{a}^t \mathbf{x} \leq \beta$ ,  $-\mathbf{a}^t \mathbf{x} \leq -\beta$ . Thus, with the observations above, it suffices to show that if a set  $S \subseteq \mathbb{R}^{k+1}$  has a description of the form

$$S = \left\{ \begin{pmatrix} \mathbf{x} \\ x_{k+1} \end{pmatrix} \in \mathbb{R}^{k+1} : \mathbf{a}_i^t \mathbf{x} + a_{i,k+1} x_{k+1} \leq b_i \quad (1 \leq i \leq m) \right\},$$

then the projection of  $S$  to  $\pi(S) \subseteq \mathbb{R}^k$  (by “deleting the last coordinate”) has a representation of the same type. We may assume that the inequality system has been ordered so

that

$$\begin{aligned} a_{i,k+1} &> 0 && \text{for } 1 \leq i \leq i_0, \\ a_{j,k+1} &< 0 && \text{for } i_0 < j \leq j_0, \\ a_{i,k+1} &= 0 && \text{for } j_0 < i \leq m. \end{aligned}$$

Now for any given  $\mathbf{x} \in \mathbb{R}^k$ , it is easy to decide whether it lies in  $\pi(S)$ . Namely,  $\mathbf{x} \in \pi(S)$  holds if and only if there is some value  $\xi \in \mathbb{R}$  such that  $\begin{pmatrix} \mathbf{x} \\ \xi \end{pmatrix} \in S$ , where the inequalities for  $1 \leq i \leq i_0$  provide upper bounds for such a value  $\xi$ , the inequalities for  $i_0 < j \leq j_0$  give lower bounds, the others provide no conditions. Thus the system has a solution  $\xi$  for given  $\mathbf{x}$  if all the upper bounds are at least as large as all the lower bounds. Explicitly, this yields a description of  $\pi(S)$  as

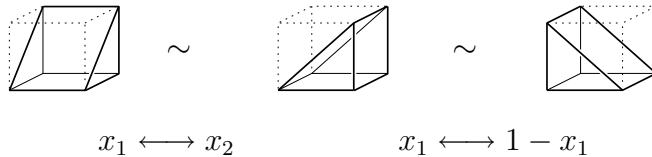
$$\left\{ \begin{array}{l} \mathbf{x} \in \mathbb{R}^k : (a_{i,k+1}\mathbf{a}_j - a_{j,k+1}\mathbf{a}_i)^t \mathbf{x} \leq a_{i,k+1}b_j - a_{j,k+1}b_i \quad \text{for } 1 \leq i \leq i_0 \\ \hspace{10em} \text{and } i_0 < j \leq j_0, \\ a_i \mathbf{x} \leq b_i \quad \text{for } j_0 < i \leq m \end{array} \right\},$$

which is a presentation of the required form.  $\square$

The transformation of an inequality system for  $S$  into a system for  $\pi(S)$  in this way is known as *Fourier-Motzkin elimination* of the last variable [58, Lecture 1]. Note that, in the worst case, the system for  $\pi(S)$  may have as many as  $\left(\frac{m}{2}\right)^2$  inequalities: much more than the system for  $S$ ! The good news at this point is that the inequality descriptions of  $\pi(S)$  are typically very redundant: many of the inequalities can be deleted without changing the set of solutions of the system. However, the bad news is that even a minimal system — which in the case of a full-dimensional polytope  $P$  consists of exactly one inequality for each facet of  $P$  — may be huge. Correspondingly, 0/1-polytopes with rather few vertices may have “many” facets: See Section 2 below.

A projection argument together with the basic operation of “switching” will allow us for the following to assume that the polytopes under consideration are full-dimensional, and have  $\mathbf{0}$  as a vertex, whenever that seems convenient:

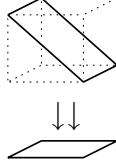
- (1) All the symmetries of the 0/1-cube  $C_d = [0, 1]^d$  transform 0/1-polytopes into 0/1-polytopes. In coordinates, these symmetries are generated by
  - permuting coordinates, and
  - replacing some coordinates  $x_i$  by  $\bar{x}_i := 1 - x_i$  (*switching*).



We call two 0/1-polytopes  $P$  and  $P'$  *0/1-equivalent* if a sequence of such operations can transform  $P$  into  $P'$ . In particular, one can transform any 0/1-polytope  $P$  with a vertex  $\mathbf{v} \in P \cap \{0, 1\}^n$  to a new, 0/1-equivalent polytope  $P'$  such that the vertex  $\mathbf{v}$  gets mapped to the vertex  $\mathbf{0}$  of  $P'$ .



- (2) If  $P \subseteq \mathbb{R}^{d+1}$  is not *full-dimensional*, then it is affinely equivalent to a 0/1-polytope  $P' \subseteq \mathbb{R}^d$ . To see this, first we may assume that  $\mathbf{0} \in P$  (after switching), so  $P$  satisfies an equation of the form  $\mathbf{a}^t \mathbf{x} + a_{d+1} x_{d+1} = 0$  with  $\mathbf{a} \in \mathbb{R}^d$ .



Furthermore, after permuting the coordinates we get that  $a_{d+1} \neq 0$ . But then “deleting the last coordinate”

$$\pi : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$$

projects  $P \rightarrow P' = \pi(P)$  injectively, that is, it defines an affine equivalence between  $P$  and  $\pi(P) = P'$ .

In the following, we usually deal with full-dimensional 0/1-polytopes, and we take 0/1-equivalence as the basic notion for their comparison. The resulting classification is much finer than the classification by affine equivalence — for example, all  $d$ -dimensional 0/1-simplices are affinely equivalent, but they are not necessarily 0/1-equivalent: Note that 0/1-equivalent polytopes are congruent, so they have the same edge lengths, volumes, etc. But the converse is not true, see below.

### Definition 6

The *faces* of a 0/1-polytope  $P$  are the subsets of the form  $P^c = \{\mathbf{x} \in P : \mathbf{c}^t \mathbf{x} = \gamma\}$ , where  $\mathbf{c}^t \mathbf{x} \leq \gamma$  is a linear inequality that is valid for *all* points of  $P$ . This definition of faces includes the subsets  $\emptyset$  and  $P$ , the *trivial faces* of  $P$ .

All faces of a 0/1-polytope are themselves 0/1-polytopes, of the form  $F = \text{conv}(F \cap \{0, 1\}^d)$ . The set of 0-dimensional faces, or *vertices*, of a 0/1-polytope is given by  $V = P \cap \mathbb{Z}^d$ . The 1-dimensional faces are called *edges*. Vertices and edges together form the *graph* of the polytope. The maximal non-trivial faces, of dimension  $\dim(P) - 1$ , are the *facets* of  $P$ . These are essential for the  $\mathcal{H}$ -presentation of polytopes: In the full-dimensional case every irredundant  $\mathcal{H}$ -presentation consists of exactly one inequality for each facet of  $P$ .

The *face lattice* is the set of all faces of  $P$ , partially ordered by inclusion. It is a graded lattice of length  $\dim(P) + 1$ . Two polytopes are *combinatorially equivalent* if their face lattices are isomorphic as finite lattices.

### Proposition 7

On the finite set of all 0/1-polytopes in  $\mathbb{R}^d$  one has the following hierarchy of equivalence relations:

“0/1-equivalent”  $\Rightarrow$  “congruent”  $\Rightarrow$  “affinely equivalent”  $\Rightarrow$  “combinatorially equivalent.”

For all three implications the converse is false, even when we restrict the discussion to full-dimensional polytopes.

**Proof.** The hierarchy is clearly valid: Every 0/1-equivalence is a congruence, congruent polyhedra are affinely equivalent, and affine equivalence implies combinatorial equivalence. In the following we provide counterexamples for all the converse implications.

(1) Full-dimensional 0/1-polytopes that are congruent but not 0/1-equivalent can be found in dimension 5:

VERTICES	VERTICES
1 0 0 0 0 0	1 0 0 0 0 0
1 0 0 1 1 0	1 0 0 1 1 0
1 0 1 0 1 0	1 0 1 0 1 0
1 1 0 0 1 0	1 0 1 1 0 0
1 0 1 1 0 0	1 1 0 0 1 0
1 0 1 1 0 1	1 1 0 0 1 1

One easily checks that these two data sets (in `polymake` input format; see `CNG:5-6a.poly` and `CNG:5-6b.poly` in the `polymake` database) describe congruent, full-dimensional 0/1-simplices in  $\mathbb{R}^5$ : For this one just computes the pairwise distances of the points. A 0/1-equivalence would transform the array on the left to the array on the right by permuting rows and columns, and by complementing columns. But on the left we have two columns with exactly one 1 (and no column with five 1s), while on the right there is only one column with exactly one 1 (and no column with five 1s). Thus the two simplices are not 0/1-equivalent. Volker Kaibel has additionally shown that for  $d \leq 4$  all congruent full-dimensional 0/1-polytopes are indeed 0/1-equivalent.

(2) The above classification for  $d = 3$  contains examples of tetrahedra that are not congruent, but of course affinely equivalent. Further examples will appear in Lecture 2.

(3) Here are two 5-polytopes, `EQU:5-7a.poly` and `EQU:5-7b.poly`, that are combinatorially, but not affinely equivalent:

VERTICES	VERTICES
1 0 0 0 0 0	1 0 0 0 0 0
1 1 0 0 0 0	1 1 1 0 0 0
1 0 1 0 0 0	1 0 1 1 0 0
1 0 0 1 0 0	1 0 0 1 1 0
1 0 0 0 1 0	1 0 0 0 1 1
1 0 0 0 0 1	1 1 0 0 0 1
1 1 1 1 1 1	1 1 1 1 1 1

In fact, each of them is a bipyramid over a 4-simplex (and hence they are combinatorially equivalent), but in the first one the main diagonal is divided in the ratio 1 : 4, for the other one the ratio is 2 : 3, and such ratios are preserved by affine equivalences.  $\square$

### 1.3 Doubly exponentially many 0/1-polytopes

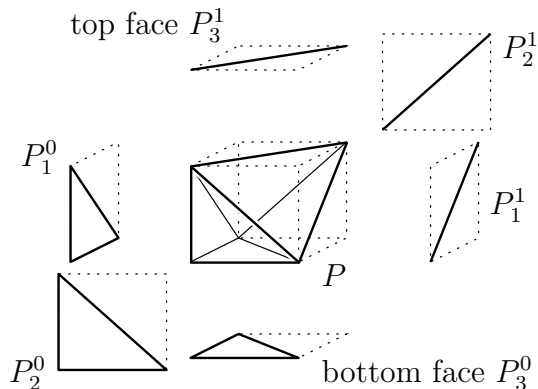
How many non-equivalent 0/1-polytopes are there? Clearly in  $\mathbb{R}^d$  there are exactly  $2^{2^d}$  different 0/1-polytopes, but some of them are low-dimensional, and some of them are equivalent to many others. Nevertheless, this trivial estimate is not that far from the truth.

For the following, let  $F_i^0$  denote the facet of the  $d$ -cube  $[0, 1]^d$  that is given by  $x_i = 0$ , and similarly let  $F_i^1$  be the facet given by  $x_i = 1$ . With a 3-dimensional picture in the back of our minds, we will refer to  $F_d^0$  as the *bottom facet* and to  $F_d^1$  as the *top facet* of  $C_d$ . All other facets will be called the *vertical facets* of  $C_d$ .

This terminology corresponds to one of the main proof techniques that we have for 0/1-polytopes: decomposition into “top” and “bottom” with induction over the dimension. For this we note the following for an arbitrary 0/1-polytope  $P \subseteq [0, 1]^d$ :

- Every facet  $F_i^s$  induces a face  $P_i^s := F_i^s \cap P$  of  $P$ ; these faces are referred to as the *trivial faces* of  $P$ .
- Every vertex of  $P$  is contained either in the *bottom face*  $P_d^0 = F_d^0 \cap P$  or in the *top face*  $P_d^1 = F_d^1 \cap P$  of  $P$ .
- Every vertex  $\mathbf{v}$  of  $P$  is determined by the set of trivial faces  $P_i^0$  that contain it, since  $v_i = 0$  holds if and only if  $\mathbf{v} \in P_i^0$ .

The following figure illustrates that in general some trivial faces are facets, while others are not.



#### Proposition 8 (Sarangerajan-Ziegler)

There is a family  $\mathcal{F}_d$  of  $2^{2^{d-1}-4}$  different, full-dimensional 0/1-polytopes in  $[0, 1]^d$ , such that

- any two polytopes in  $\mathcal{F}_d$  are 0/1-equivalent if and only if they are combinatorially equivalent, and

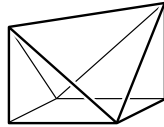
- for  $d \geq 6$ , the collection  $\mathcal{F}_d$  contains more than  $2^{2^{d-2}}$  combinatorially non-equivalent  $d$ -dimensional 0/1-polytopes in  $\mathbb{R}^d$ .

**Proof.** Let  $d \geq 3$ , and let  $\mathcal{F}_d$  be the set of 0/1-polytopes  $P(V) = \text{conv}(V)$  of the following form:

- $V$  contains all the vertices in the bottom facet  $F_d^0$  of the  $d$ -cube  $[0, 1]^d$  (that is,  $\{0, 1\}^{d-1} \times \{0\} \subseteq V$ ),
- the pair  $e_d, \mathbf{1}$  of opposite vertices of the top facet  $F_d^1$  is contained in  $V$ ,
- the two opposite vertices  $e_d + e_1, \mathbf{1} - e_1$  of the top facet  $F_d^1$  are *not* contained in  $V$ .

This fixes  $2^{d-1} + 4$  vertices to be inside or outside  $V$ , and thus leaves  $2^{2^d - (2^{d-1} + 4)} = 2^{2^{d-1} - 4}$  choices for the set  $V$ , and hence for the polytope  $P(V)$ .

For  $d = 3$ , there is exactly one polytope of the given special type (the “nameless” one):



Now the following facts are easy to verify about the polytopes  $P(V) \in \mathcal{F}_d$ :

- $P(V)$  is a  $d$ -dimensional 0/1-polytope. Its bottom facet  $P_d^0 = F_d^0$  is a  $(d - 1)$ -cube, with  $2^{d-1}$  vertices.
- All the vertical facets  $F_i^s$  ( $i < d$ ) induce facets  $P_i^s$  of  $P(V)$ . These are the facets of  $P(V)$  that are adjacent to the cube facet  $P_d^0$ . Every vertex of  $P(V)$  that is not on  $F_d^0$  is completely determined by the set of vertical facets  $P_i^s$  that it lies on.
- All facets of  $P(V)$ , other than the bottom facet, have fewer than  $2^{d-1}$  vertices. (For this we use that only the  $2d + 2\binom{d}{2} = d^2 + d$  “special” hyperplanes given by  $x_i = 0$ ,  $x_i = 1$  or  $x_i = x_j$  or  $x_i = 1 - x_j$  contain  $2^{d-1}$  0/1-points, and all other hyperplanes contain less than  $2^{d-1}$  0/1-points. It is easy to verify that no special hyperplane other than “ $x_d = 0$ ” can describe a facet of  $P(V)$ .)
- Therefore, if two polytopes  $P(V)$  and  $P(V')$  are combinatorially isomorphic, then they are equivalent by a symmetry of the  $d$ -dimensional 0/1-cube that fixes the bottom facet, and induces an automorphism of that bottom facet.
- The order of the symmetry group of  $C_{d-1}$  is  $2^{d-1}(d-1)!$ . So for each  $P(V)$  there are not more than  $2^{d-1}(d-1)!$  polytopes  $P(V')$  that are combinatorially equivalent to it.
- Therefore, there are more than  $2^{2^{d-1}-4}/(2^{d-1}(d-1)!)$  combinatorially non-isomorphic 0/1-polytopes of the form  $P(V)$ , and for  $d > 5$  this number is larger than  $2^{2^{d-2}}$ .  $\square$

## 2 The Number of Facets

### 2.1 Some examples

Staring too much at the 3-dimensional case, one might come up with the conjecture that a  $d$ -dimensional 0/1-polytope cannot have more than  $2^d$  facets. In fact,

$$C_d^\Delta := \text{conv}\{\mathbf{e}_1, \dots, \mathbf{e}_d, \mathbf{1} - \mathbf{e}_1, \dots, \mathbf{1} - \mathbf{e}_d\}$$

is a polytope with  $2d$  vertices ( $d \geq 3$ ) that is centrally symmetric with respect to  $\frac{1}{2}\mathbf{1}$ , the center of the 0/1-cube. Hence it is affinely equivalent to the usual regular  $d$ -dimensional cross polytope. In particular, this polytope has  $2^d$  facets. The first examples are given as `CR0:3-6.poly`, `CR0:4-8.poly`, ... in the database.

(For  $d = 4$  this construction produces a regular cross polytope `CR0:4-8.poly`, all of whose edges have length  $\sqrt{2}$ . Another remarkable regular cross polytope `HAM:8-16.poly` arises from the extended Hamming code  $\tilde{H}_8$ . The cross polytopes  $C_d^\Delta$  as constructed above are not regular for  $d \neq 4$ : they have edges of lengths  $\sqrt{2}$  and  $\sqrt{d-2}$ .)

But more than that? Evgenij Gawrilow was the first to detect a 5-dimensional 0/1-polytope with 40 facets. After intensive search, here is what we know about examples of low-dimensional 0/1-polytopes with “many facets” — and thus about  $\#f(d)$ , the maximal number  $f_{d-1}(P)$  of facets that a  $d$ -dimensional 0/1-polytope  $P$  can have:

$d$	$\#f(d)$	proved/found by	example
3	= 8		<code>CR0:3-6.poly</code>
4	= 16	Below	<code>CR0:4-8.poly</code>
5	= 40	Aichholzer	<code>EG:5-10.poly</code>
6	$\geq 121$	Sarangarajan	<code>AS:6-18.poly</code>
7	$\geq 432$	Christof	<code>TC:7-30.poly</code>
8	$\geq 1675$	Christof	<code>TC:8-38.poly</code>
9	$\geq 6875$	Christof	<code>TC:9-48.poly</code>
10	$\geq 41591$	Christof	<code>TC:10-83.poly</code>
$\vdots$		$\vdots$	$\vdots$
13	$\geq 17464356$	Christof	<code>TC:13-254.poly</code>

In brief: 0/1-polytopes may have *many* facets. But how many, at most? And how do 0/1-polytopes with “many facets” look like?

### 2.2 Some upper bounds

The asymptotically best upper bound for the number of facets of a  $d$ -dimensional 0/1-polytope is the following. I assume that it is rather tight; the problem is with the lower bounds, which look much worse.

**Theorem 9 (Fleiner, Kaibel & Rote [26])**

For all large enough  $d$ , a  $d$ -dimensional 0/1-polytope has no more than

$$\#f(d) \leq 30(d-2)!$$

facets.

See [26] for the (beautiful) proof of this result, which is probably valid for *all*  $d$ . The first bound of this order of magnitude was pointed out by Imre Bárány [58, p. 26]. Here we present a proof for the inequality

$$\#f(d) \leq 2(d-1)! + 2(d-1), \tag{*}$$

which is asymptotically a bit worse than the one just quoted, but it is better in low dimensions — and whose proof (also from [26]) is strikingly simple.

For this, let  $P \subseteq [0, 1]^d$  be a  $d$ -dimensional 0/1-polytope. We note the following facts:

- The volume  $\text{Vol}_d(P)$  is an integral multiple of  $\frac{1}{d!}$ .  
(Every polytope can be triangulated without new vertices. Thus we are reduced to the case of 0/1-simplices, whose volume is given as  $\frac{1}{d!}$  times the determinant — which is an integer.)
- The number of facets  $f_{d-1}(P)$  of a  $d$ -dimensional 0/1-polytope  $P$  satisfies

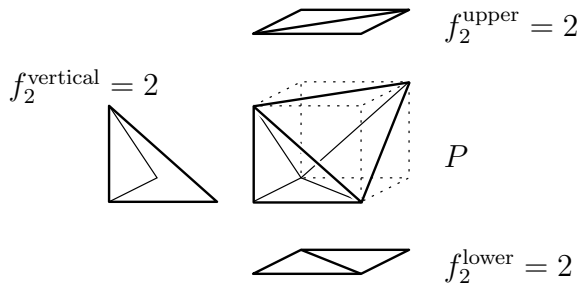
$$f_{d-1}(P) \leq 2d + d!(1 - \text{Vol}_d(P)).$$

(This follows from an observation of Bárány: The  $d$ -cube  $[0, 1]^d$  has  $2d$  facets. Now delete the “superfluous” 0/1-vectors, so that  $[0, 1]^d$  is gradually transformed into  $P$ . Whenever a facet of  $P$  “appears” in this process, a pyramid over the facet is removed, and the volume of this pyramid is at least  $\frac{1}{d!}$ .)

- Consider the projection  $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$  that deletes the last coordinate. With respect to this projection, the boundary of  $P$  may be divided into “vertical,” “upper” and “lower” facets. After projection, the images of the upper facets partition  $\pi(P)$  into  $(d-1)$ -dimensional 0/1-polytopes, and so do the lower facets. Thus we get that

$$f_{d-1}^{\text{upper}}(P), f_{d-1}^{\text{lower}}(P) \leq (d-1)! \text{Vol}_{d-1}(\pi(P)).$$

Our figure illustrates this decomposition for  $d = 3$ :



- At the same time, the vertical facets of  $P$  are in bijection with a subset of the facets of  $\pi(P)$ : and the number of these can be estimated using the formula above:

$$f_{d-1}^{\text{vertical}}(P) \leq f_{d-2}(\pi(P)) \leq 2(d-1) + (d-1)!(1 - \text{Vol}_{d-1}(\pi(P))).$$

- ... and summing the upper bounds that we have obtained for  $f_{d-1}^{\text{upper}}(P)$ ,  $f_{d-1}^{\text{lower}}(P)$  and  $f_{d-1}^{\text{vertical}}(P)$  completes the proof of (\*).  $\square$

## 2.3 A bad construction

All the available data suggests that 0/1-polytopes may have much more than just simply-exponentially many facets. But no one has been able, up to now, to prove any lower bound on  $\#f(d)$  that grows faster than  $c^d$ , for some constant  $c > 1$ .

### Proposition 10 (Kortenkamp et al. [43])

For all large enough  $d$ ,

$$\#f(d) > 3.6^d.$$

**Proof.** The *sum*  $P_1 * P_2$  of two polytopes  $P_1$  and  $P_2$  is obtained by representing the polytopes in some  $\mathbb{R}^n$  in such a way that their intersection consists of one single point which for both of them lies in the relative interior, and by then taking the convex hull:

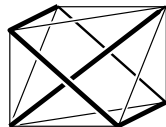
$$P := \text{conv}(P_1 \cup P_2), \quad \text{if } P_1 \cap P_2 = \{\mathbf{x}\} \text{ is a relative interior point for both } P_1 \text{ and } P_2.$$

If we take the sum of two polytopes in this way, then the dimensions add, while the number of facets are multiplied. As an example, the sum of an  $n$ -gon (dimension 2,  $n$  facets) and an interval (dimension 1, 2 facets) results in a bipyramid over the  $n$ -gon (dimension 3,  $2n$  facets). The sum operation is polar to taking products, where the dimensions add and the numbers of vertices are multiplied.

But we have to take a bit of care in order to adapt this general polytope operation to 0/1-polytopes, since there is very little space for “moving into a position” if we want to stay within the setting of 0/1-polytopes. For this call a 0/1-polytope *centered* if it has the center point  $\frac{1}{2}\mathbf{1}$  in its (relative) interior. For example, among the 3-dimensional 0/1-polytopes, the 3-dimensional prism, the two different pyramids over a square, and the tetrahedra except for CUT(3), are *not* centered! On the other hand, the cross polytopes  $C_d^\Delta$  are centered for all  $d$ .

The sum of two centered 0/1-polytopes  $P_1 \subseteq [0, 1]^{d_1}$  and  $P_2 \subseteq [0, 1]^{d_2}$  can be realized in  $[0, 1]^{d_1+d_2}$  by embedding them into the subspaces  $x_{d_1} = x_{d_1+1} = \dots = x_{d_1+d_2}$  resp.  $x_1 = \dots = x_{d_1} = 1 - x_{d_1+1}$ . This yields centered 0/1-polytopes  $\widehat{P}_1, \widehat{P}_2 \subseteq [0, 1]^{d_1+d_2}$  that are affinely isomorphic to  $P_1$  and  $P_2$ , and whose convex hull realizes the sum  $P_1 * P_2$ . For

example, the octahedron  $C_3^\Delta$  can be viewed as the sum of a rectangle and a diagonal:



Now we need a starting block: and for that we use Christof's 13-dimensional 0/1-polytope `TC:13-254.poly` with at least  $17464356 > 3.6^{13}$  facets. This polytope is indeed centered (you may check that already the first 22 vertices contain  $\frac{1}{2}\mathbf{1}$  in their interior). Taking sums of copies of this polytope, and extra copies of  $[0, 1]$  if needed, we arrive at the result.  $\square$

This seems to be the best asymptotic lower bound available in the moment. I think that it is *bad*: one should be able to prove a lower bound of the form  $c^{d \log d}$ , or at least that there is a lower bound that grows faster than  $C^d$  for every  $C > 1$ ! I'd offer two candidates for such a lower bound construction: Random polytopes, and cut polytopes. However, we cannot do the corresponding lower bound proof for either of these two classes, up to now.

### 3 Random 0/1-Polytopes

We do not understand random 0/1-polytopes very well. Let  $d$  be not too small, and take, say,  $2d$  or  $d \log d$  or  $d^2$  random 0/1-points: *How will their convex hull look like? How many edges, and how many facets can we expect the random polytope to have?* We will see in this section that the analysis of random 0/1-polytopes is driven by one basic linear algebra parameter: the probability  $P_d$  that a random 0/1-matrix of size  $d \times d$  has vanishing determinant.

This probability corresponds to the case of  $d + 1$  random 0/1-points: Take  $d + 1$  points  $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_d$  independently at random (where all 0/1 points appear with the same probability  $p = \frac{1}{2^d}$ ). The  $d+1$  points will be distinct with very high probability, and by symmetry we may assume that the first point is  $\mathbf{v}_0 = \mathbf{0}$ . Thus the probability that the  $d + 1$  points span a  $d$ -dimensional simplex is exactly  $1 - P_d$ . How large is this probability? We first study the case where  $d$  is small, and from this we will derive a quite misleading impression.

#### 3.1 The determinant of a small random 0/1-matrix

Let  $P_d$  be the probability that a random 0/1-matrix of size  $d \times d$  is singular. Of course we have

$$P_d = \frac{M_d}{2^{d^2}},$$



where  $M_d$  denotes the number of different 0/1-matrices of size  $d \times d$  that have determinant 0. This number can be computed explicitly for  $d \leq 7$ :

$d$	$M_d$
1	1
2	10
3	338
4	42976
5	21040112
6	39882864736
7	292604283435872

(In fact, for  $d \leq 6$  numbers that are equivalent to these were computed by Mark B. Wells in the sixties [44, p. 198]; the value for  $d = 7$  is new, due to Gerald Stein.)

From this, we get a table for  $P_d$ , where for  $d \geq 8$  we print estimates that were obtained by taking 10 million random matrices for each case:

$$\begin{aligned}
 P_1 &= 0.5 \\
 P_2 &= 0.625 \\
 P_3 &= 0.66015625 \\
 P_4 &= 0.65576\dots \\
 P_5 &= 0.62704\dots \\
 P_6 &= 0.58037\dots \\
 P_7 &= 0.51976\dots \\
 P_8 &\approx 0.449 \\
 P_9 &\approx 0.373 \\
 P_{10} &\approx 0.298 \\
 P_{11} &\approx 0.226 \\
 P_{12} &\approx 0.164 \\
 P_{13} &\approx 0.113 \\
 P_{14} &\approx 0.075 \\
 P_{15} &\approx 0.047
 \end{aligned}$$

Conclusion: the probability  $P_d$  first increases (!), but then it seems to decrease and approach 0 steadily, but not very fast.

### 3.2 Komlós' theorem and its consequences

The question about the probability  $P_d$  of singular random 0/1-matrices is equivalent to the same question about  $\pm 1$ -matrices:  $P_d$  is equally the probability that a random  $\pm 1$ -matrix of size  $(d+1) \times (d+1)$  is singular. It is often convenient to switch to the  $\pm 1$ -case because it has more symmetry. The following proposition establishes the equivalence. Its observation is quite trivial, but also fundamental for various problems related to 0/1-polytopes.

**Proposition 11 (Williamson [57])**

The map

$$\varphi: A \mapsto \begin{pmatrix} 1 & \mathbf{1}^t \\ \mathbf{1} & \mathbf{1}\mathbf{1}^t - 2A \end{pmatrix} =: \widehat{A}.$$

establishes a bijection between the 0/1-matrices of size  $d \times d$  and the  $\pm 1$ -matrices of size  $(d+1) \times (d+1)$  for which all entries in the first row and column are +1.

The bijection  $\varphi$  satisfies  $\det(\widehat{A}) = (-2)^d \det(A)$ . In particular, it also provides a bijection between the invertible matrices of the two types.

Furthermore, there is a one-to- $2^{2d+1}$  correspondence between the 0/1-matrices of size  $d \times d$  and the  $\pm 1$ -matrices of size  $(d+1) \times (d+1)$ . The correspondence again respects invertibility.

**Proof.** Geometrically, the map  $\varphi$  realizes an embedding of  $[0, 1]^d$  as a facet of  $[-1, +1]^{d+1}$ . Algebraically,  $\widehat{A} = \begin{pmatrix} 1 & \mathbf{0}^t \\ \mathbf{1} & I_d \end{pmatrix} \begin{pmatrix} 1 & \mathbf{1}^t \\ \mathbf{0} & -2A \end{pmatrix}$  arises from  $\begin{pmatrix} 1 & \mathbf{1}^t \\ \mathbf{0} & -2A \end{pmatrix}$  by adding the first row to all others, and thus we see that  $\varphi(A)$  is indeed invertible if  $A$  is, and that  $\det(\widehat{A}) = (-2)^d \det(A)$ .

Finally, with every  $\pm 1$ -matrix one can associate a canonical matrix of the same size and type for which the first row and column are positive: for this first multiply columns by  $-1$  in order to make the first row positive, then multiply rows to make the first column positive. There are exactly  $2^{2d+1}$  matrices in  $\{-1, +1\}^{(d+1) \times (d+1)}$  that have the same canonical form, corresponding to the  $2d + 1$  entries in the first row and column for which a sign can be chosen.  $\square$

Thus  $P_d$  measures for 0/1-matrices as well as for  $\pm 1$ -matrices the probability of determinant 0. Our experimental evidence is that  $P_d$  should converge to 0. But how fast? Here is what we know.

**Theorem 12 (Komlós' Theorem; Kahn, Komlós and Szemerédi [40])**

The probability  $P_d$  that a random 0/1-matrix of size  $d \times d$  is singular satisfies

$$\frac{d^2}{2^d} < P_d < 0.999^d$$

for all high enough  $d$ .

**Proof.** The non-trivial part is the upper bound, which is due to Kahn, Komlós and Szemerédi [40]. Their proof is difficult, involving a probabilistic construction. In fact, it is hard enough to prove that  $P_d$  converges to zero at all: this was first proved by Komlós in 1967 [42]; good starting points are Komlós' proof for  $\lim_{d \rightarrow \infty} P_d = O(\frac{1}{\sqrt{d}})$  given in [12, Sect. XIV.2], and Odlyzko's paper [50].)

Here we only prove the lower bound. For this, we work in the  $\pm 1$ -model, where  $P_d$  denotes the probability that a random  $(d+1) \times (d+1)$ -matrix is singular. In this model, the

probability that two given rows are “equal or opposite” is  $\frac{1}{2^d}$ , and the same for two given columns. Altogether there are  $2\binom{d+1}{2} = d^2 + d$  such events. These are not independent, but for any two such events the probability that they *both* occur is at most  $\frac{1}{2^{2d-1}}$ : if we look at two events that both refer to rows, or both refer to columns, then the probability that they both occur is  $(\frac{1}{2^d})^2$ ; if we want that two specific rows are equal or opposite, and two columns are equal or opposite at the same time, then the probability is  $(\frac{1}{2^d})(\frac{1}{2^{d-1}})$ . Thus we may estimate

$$P_d \geq (d^2 + d)\frac{1}{2^d} - \binom{d^2 + d}{2}\frac{1}{2^{2d-1}}$$

and this is larger than  $\frac{d^2}{2^d}$  for  $d > 10$ . □

It has been conjectured that the lower bound of this theorem is close to the truth:

**Conjecture 13** (see [50], [40])

*The probability  $P_d$  that a random 0/1-matrix is zero is dominated by the possibility that one of the rows or columns is zero, or that two rows are equal, or two columns are equal:*

$$P_d \sim 2\binom{d+1}{2}\frac{1}{2^d} \sim \frac{d^2}{2^d}.$$

*Equivalently: if a random  $\pm 1$ -matrix of size  $(d+1) \times (d+1)$  is singular, then “most probably” two rows or two columns are equal or opposite.*

### 3.3 High-dimensional random 0/1-polytopes

Now we try to describe random 0/1-polytopes for large  $d$ .

**Corollary 14**

*With a probability that tends to 1 for  $d \rightarrow \infty$  the following is true:*

- (i) *Any polynomial number of 0/1-vectors chosen (independently, with equal probability) from  $\{0, 1\}^d$  will be distinct.*
- (ii) *A set of  $d$  randomly chosen 0/1-points spans a hyperplane that does not contain the origin  $\mathbf{0}$ .*
- (iii) *The convex hull of  $d+1$  random 0/1-points is a  $d$ -dimensional simplex.*

**Proof.** The probability for  $n$  random 0/1-vectors to be distinct is

$$\left(1 - \frac{1}{2^d}\right) \left(1 - \frac{2}{2^d}\right) \cdots \left(1 - \frac{n-1}{2^d}\right) > \left(1 - \frac{n}{2^d}\right)^n = \exp\left[n \ln\left(1 - \frac{n}{2^d}\right)\right],$$

and for  $n \ll 2^d$  this can be estimated with  $\ln(1 - \frac{n}{2^d}) \approx -\frac{n}{2^d}$ , so we get a probability of at least  $\exp(-\frac{n^2}{2^d})$ , which converges to 1 if  $\frac{n^2}{2^d}$  tends to zero.

If we choose  $d + 1$  random points, then by symmetry we may assume that the first one is  $\mathbf{0}$ . Thus the probability in question for the third statement, and also for the second one, is exactly  $1 - P_d$ , and thus both statements follow from Komlós' theorem.  $\square$

But one would like to ask more questions. For example: *What is the expected volume of a random simplex?* It is indeed huge, as one can see from the following observations of Szekeres & Turán [56], see Exercise 9: for a random 0/1-matrix  $A$  of size  $d \times d$ , the expected value for the squared determinant is exactly<sup>2</sup>

$$E(\det(A)^2) = \frac{(d+1)!}{2^{2d}}.$$

But that means that 0/1-matrices  $A$  of determinant

$$|\det(A)| \geq \frac{\sqrt{(d+1)!}}{2^d}$$

exist, and are in fact common ("to be expected"). This is to be compared with the Hadamard upper bound

$$\det(\hat{A}) \leq \sqrt{d+1}^{d+1}, \quad \det(A) \leq \frac{\sqrt{d+1}^{d+1}}{2^d}.$$

that we will meet in Section 5.2.

**Proposition 15 (Füredi [27])**

*For any constant  $\varepsilon > 0$ , a random 0/1 polytope with  $n \geq (2 + \varepsilon)d$  vertices contains  $\frac{1}{2}\mathbf{1}$ , while a random polytope with  $n \leq (2 - \varepsilon)d$  vertices does not contain  $\frac{1}{2}\mathbf{1}$ , with probability tending to 1 for  $d \rightarrow \infty$ .*

Füredi's proof is elementary, combining Komlós' theorem with an estimate about the maximal number of regions in an arrangement of hyperplanes. Perhaps it can be adapted to prove that a random 0/1 polytope with  $n \geq (2 + \varepsilon)d$  vertices should even be centered?

Another question linked to Corollary 14 is: *Can we expect that there will be further 0/1-points on the hyperplane spanned by  $d$  random points?* We don't quite know, but the following result points towards an answer.

**Proposition 16 (Odlyzko [50])**

*With probability tending to 1 for  $d \rightarrow \infty$ , and*

$$n \leq d - \frac{10d}{\log d},$$

*$n$  random 0/1-points span an affine subspace of dimension  $n$  that does not contain any further 0/1-point.*

---

<sup>2</sup>The expected value for  $\det(A)$  is 0 if  $d > 0$ , for symmetry reasons.

One interesting question is whether this result could be extended to much bigger  $n$ . Of course, by Corollary 14(iii) to Komlós' theorem the statement fails (badly) if  $n = d+1$ , but what about  $n = d$ ? In other words, is there a high probability that  $d$  random 0/1-points will span a “simplex hyperplane”?

Still another, related question is: *If  $d$  random points span a hyperplane, is there a reasonable chance that this hyperplane is very unbalanced, with only few 0/1-points on one side?* This is closely linked (by “linearity of expectation”) to the expected number of facets of a random polytope.

**Proposition 17**

*There is a constant  $c > 0$  such that a random 0/1-polytope  $P \subseteq [0, 1]^d$  with  $n \leq (1 + c)d$  vertices is “uniform” in the sense that any  $d + 1$  points span a  $d$ -simplex, with probability tending to 1 for  $d \rightarrow \infty$ .*

*(In particular, uniform polytopes are simplicial.)*

**Proof.** Let  $\gamma < 1$  be a constant such that  $P_d \leq \gamma^d$  holds for all large enough  $d$ . The probability that all  $(d + 1)$ -subsets of a random sequence of  $n$  0/1-vectors span  $d$ -simplices is at least

$$\text{Prob}(P \text{ uniform}) \geq 1 - \binom{n}{d+1} P_d > 1 - \binom{(1+c)d}{d+1} \gamma^d$$

and with  $(cd)! \approx \left(\frac{cd}{e}\right)^{cd}$  we estimate

$$\binom{(1+c)d}{d+1} \gamma^d \leq \frac{((1+c)d)^{cd}}{(cd-1)!} \gamma^d \approx \left(\frac{e(1+c)}{c}\right)^{cd} \gamma^d.$$

Thus  $\text{Prob}(P \text{ uniform})$  will tend to 1 for large  $d$  if

$$\left(\frac{e(1+c)}{c}\right)^c < \frac{1}{\gamma}.$$

Thus by Theorem 12 one can take  $c = 0.00009$ . However, if Conjecture 13 were true, then one could indeed take  $c = 0.27$ . □

Note that if  $P$  is simplicial, then  $P_1^s$  is a simplex of dimension at most  $d - 1$  for  $s = 0, 1$ , and thus  $P$  has not more than  $2d$  vertices. And simplicial polytopes with  $2d$  vertices do indeed exist: but the only examples that we know are centrally-symmetric cross polytopes, which one gets as

$$\text{conv}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d, \mathbf{1} - \mathbf{v}_1, \mathbf{1} - \mathbf{v}_2, \dots, \mathbf{1} - \mathbf{v}_d\},$$

where  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d \in \{0, 1\}^d$  are  $d$  affinely independent points whose last coordinate is 0. Are there any other examples? This is not clear, but one may note that if  $P$  is a  $d$ -dimensional cross polytope, then it must be centrally symmetric. In fact, if  $\mathbf{v}, \mathbf{w}$  are vertices of  $P$  that are not adjacent, then they are not both contained in any trivial face  $P_i^s$  (since these faces are simplices), hence they are opposite to each other in the  $d$ -cube. But is every simplicial  $d$ -dimensional 0/1-polytope with  $2d$  vertices necessarily a cross polytope?

## 4 Cut Polytopes

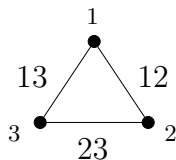
The “special” 0/1-polytopes studied in combinatorial optimization exhibit enormous complexity. One well-studied instance is that of the symmetric and asymmetric travelling salesman (TSP) polytopes (see [31]), for which Billera and Sarangarajan [9] have recently shown that *all* 0/1-polytopes appear as faces.

In this lecture, we discuss basic properties of a different family of 0/1-polytopes, the cut polytopes, and of the correlation polytopes (a.k.a. boolean quadric polytopes), which are affinely equivalent to them. For all of this and much more, Deza & Laurent [19] provides an excellent and comprehensive reference.

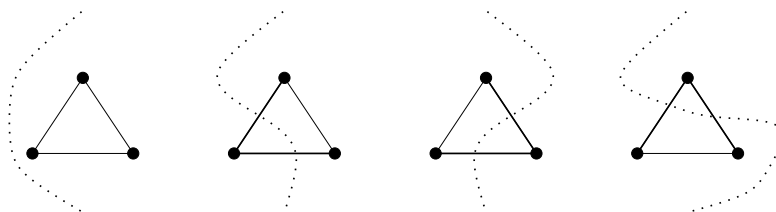
### 4.1 “Small” cut polytopes

Let’s start with a “construction by example” of the “very small” cut polytopes; the general prescription will come in the next section.

A *cut* in a graph is any edge set of the form  $E(S, V \setminus S) = E(V \setminus S, S)$ , for  $S \subseteq V$ . That is, a cut consists of all edges that connect a node in  $S$  to a node not in  $S$ . For example, the complete graph  $K_3$



has four cuts: all the edge sets of size 2, as well as the empty set of edges:



These cuts can be encoded by their *cut vectors*

$$\begin{pmatrix} x_{12} \\ x_{13} \\ x_{23} \end{pmatrix} \in \{0, 1\}^3,$$

where the  $ij$ -coordinate records whether the edge  $ij$  is in the cut or not. The cut polytope is the convex hull of all these cut vectors. So, for  $K_3$  we get the cut polytope `CUT3:3-4.poly` as

$$\text{CUT}(3) = \text{conv} \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

This 0/1-polytope is the convex hull of all 0/1-vectors of even weight (those just happen to be the cuts), so it is the regular simplex of side-length  $\sqrt{2}$ . Not a very interesting 0/1-polytope.

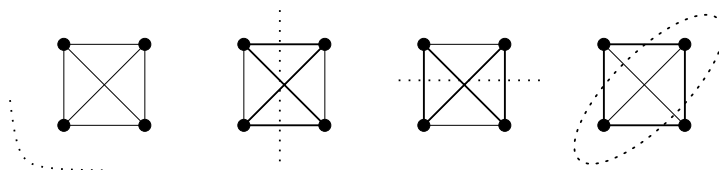
The complete graph  $K_4$  has  $\binom{4}{2} = 6$  edges, and altogether 8 cuts: the empty cut, the four cuts of size 3 that separate one vertex from the three others, and three cuts of size 4 that separate two vertices from the two others. Each cut yields a cut vector

$$(x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34})^t \in \{0, 1\}^6.$$

The resulting polytope `CUT4:6-8.poly` again has a very simple structure: it is a sum of two simplices,

$$\text{CUT}(4) \cong \text{CUT}(3) * \text{CUT}(3) \cong \Delta_3 * \Delta_3.$$

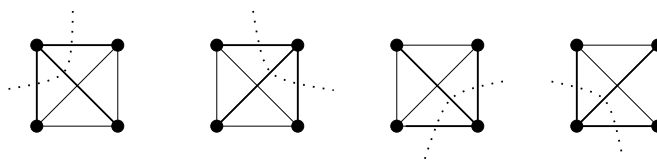
To see this, note that four of the eight cuts of  $K_4$



contain “none or both” from each pair of disjoint edges in  $K_4$ , that is,

$$x_{12} = x_{34}, \quad x_{13} = x_{24}, \quad x_{14} = x_{23},$$

so they lie in the 3-dimensional subspace  $U_1$  of  $[0, 1]^6 \subseteq \mathbb{R}^6$  that is given by these three equations. The other four cuts (of size 3)



all contain exactly one edge from each disjoint pair, that is, they lie in the 3-dimensional subspace  $U_2$  given by

$$x_{12} + x_{34} = 1, \quad x_{13} + x_{24} = 1, \quad x_{14} + x_{23} = 1$$

and give a 3-simplex that is equivalent to  $\text{CUT}(3)$  in this subspace. Now  $U_1 \cap U_2 = \{\frac{1}{2}\mathbf{1}\}$  completes the analysis: we *understand the structure* of  $\text{CUT}(4)$ . (Combinatorially,  $\text{CUT}(4)$  may also be identified with the cyclic polytope  $C_6(8)$ ; in particular, it is simplicial, and neighborly. But nevertheless, it is not a very interesting polytope.)

And so on ... ? It turns out that the cut polytopes are much more complicated (“interesting”) than one might think.

## 4.2 Cut polytopes and correlation polytopes

The definition/construction of the general cut polytopes follows a general method that has proved to be extremely successful in combinatorial optimization: The cuts in a complete graph  $K_n$  are encoded into the 0/1-polytope given by their characteristic vectors.

### Definition 18 (Cut polytopes)

With every subset  $S \subseteq [n] := \{1, \dots, n\}$ , associate a 0/1-vector

$$\delta(S) \in \{0, 1\}^d, \quad d = \binom{n}{2},$$

by setting (for  $1 \leq i < j \leq n$ )

$$\delta(S)_{ij} := \begin{cases} 1 & \text{if } |S \cap \{i, j\}| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus we can identify the coordinates  $x_{ij}$  of  $\mathbb{R}^d$  with the edge set of  $K_n$  (a complete graph with vertex set  $[n]$ ), and the vector  $\delta(S)$  represents the set  $\{ij : x_{ij} = 1\}$  of edges  $ij$  of  $K_n$  that connect a vertex in  $S$  with a vertex in  $\bar{S} := [n] \setminus S$ , that is, a *cut*  $E(S, \bar{S})$  in  $K_n$ .

The *cut polytope*  $\text{CUT}(n)$  is defined by

$$\text{CUT}(n) := \text{conv} \{ \delta(S) : S \subseteq [n] \} \subseteq \mathbb{R}^d.$$

### Lemma 19

For every  $n \geq 1$ , and  $d = \binom{n}{2}$ , the cut polytope  $\text{CUT}(n)$  is a centered  $d$ -dimensional polytope with  $2^{n-1}$  vertices.

**Proof.** The two sets  $S$  and  $\bar{S}$  determine the same cut  $\delta(S) = \delta(\bar{S})$ , but any two subsets  $S, S' \subseteq [n-1]$  with  $S \neq S'$  determine different cuts  $\delta(S) \neq \delta(S')$ , since  $S = \{i \in [n-1] : \delta(S)_{in} = 1\}$ . Thus

$$\text{CUT}(n) = \text{conv} \{ \delta(S) : S \subseteq [n-1] \} \subseteq \mathbb{R}^d$$

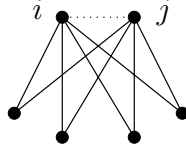
has  $2^{n-1}$  vertices (corresponding to the  $2^{n-1}$  cuts of  $K_n$ ). If  $\text{CUT}(n) \subseteq \mathbb{R}^d$  were not full-dimensional, then it would satisfy some linear equation:

$$\mathbf{a}^t \mathbf{x} = \sum_{i,j} a_{ij} x_{ij} = \beta \quad \text{for all } x \in \text{CUT}(n)$$

for some non-zero  $\mathbf{a} \in \mathbb{R}^d$ . However, the zero cut  $\delta(\emptyset) = \mathbf{0} \in \text{CUT}(n)$  yields  $\beta = 0$ . Furthermore, we derive from the sketch below that

$$\delta(\{i\}) + \delta(\{j\}) - \delta(\{i, j\}) = 2\mathbf{e}_{ij},$$





so  $\mathbf{a}^t \delta(S) = 0$  for all  $S \subseteq [n]$  implies that

$$\mathbf{a}^t(2e_{ij}) = 2a_{ij} = 0 \quad \text{for all } \{i, j\} \subseteq [n],$$

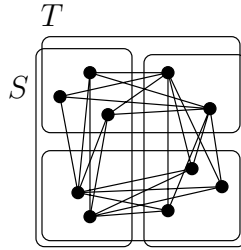
and thus  $\mathbf{a} = \mathbf{0}$ .

To see that the cut polytopes are centered, it suffices to note any edge  $ij$  will be contained in a random cut with probability exactly  $\frac{1}{2}$ . Thus the average over all vertices of  $\text{CUT}(n)$  (that is, the centroid of the set of vertices) is  $\frac{1}{2}\mathbf{1}$ .  $\square$

We note one more feature of the polytope  $\text{CUT}(n)$ : it is very symmetric, with a vertex-transitive symmetry group. In fact, every symmetric difference of two cuts is a cut: this follows from the equation

$$E(S, \bar{S}) \Delta E(T, \bar{T}) = E(S \Delta T, \overline{S \Delta T}),$$

which is best verified and visualized in a little picture such as the following:



Thus for any  $S \subseteq [n]$  the *switching map*

$$\sigma_S : \mathbb{R}^d \rightarrow \mathbb{R}^d$$

$$x_{ij} \mapsto \begin{cases} 1 - x_{ij} & \text{if } ij \in E(S, \bar{S}), \text{ i. e., if } \delta(S)_{ij} = 1, \\ x_{ij} & \text{otherwise,} \end{cases}$$

defines an automorphism of  $\text{CUT}(n)$  that takes  $\delta(T)$  to  $\delta(T \Delta S)$ , and thus takes the vertex  $\delta(S)$  to the vertex  $\delta(\emptyset) = \mathbf{0}$ , and conversely. Thus, under such switching operations all vertices of  $\text{CUT}(n)$  are equivalent!

Next we will look at a different class of important 0/1-polytopes: the cut polytopes in (thin) disguise.

**Definition 20 (Correlation polytopes)**

The  $n$ -th *correlation polytope* is the convex hull of all  $n \times n$  0/1-matrices of rank 1:

$$\text{COR}(n) := \text{conv}\{\mathbf{x}\mathbf{x}^t : \mathbf{x} \in \{0, 1\}^n\} \subseteq \mathbb{R}^{n^2}.$$

It is not so hard to see directly that  $\text{COR}(n)$  is a polytope of dimension  $\binom{n+1}{2}$  with  $2^n$  vertices, but the following observation yields even more.

**Lemma 21 (de Simone [18])**

For  $n \geq 2$  and  $d = \binom{n}{2}$ , there is a linear map

$$\gamma : \mathbb{R}^{\binom{n-1}{2}} \longrightarrow \mathbb{R}^d$$

that induces an affine isomorphism of polytopes

$$\gamma : \text{COR}(n-1) \cong \text{CUT}(n).$$

**Proof.** For every correlation matrix  $\mathbf{x}\mathbf{x}^t$  we can extract the vector  $\mathbf{x}$  from its diagonal, from this derive a set  $S_{\mathbf{x}} := \{i \in [n-1] : x_i = 1\}$ , and thus get the cut vector  $\delta(S_{\mathbf{x}})$ . Furthermore, the components of  $\delta(S_{\mathbf{x}})$  can be derived as *linear* combinations of the entries of  $\mathbf{x}\mathbf{x}^t$ :

$$\begin{aligned} \delta_{ij} &:= x_i(1-x_j) + x_j(1-x_i) = x_{ii} - x_{ij} + x_{jj} - x_{ji}, \\ \text{and } \delta_{in} &:= x_i = x_{ii}. \end{aligned}$$

This defines a linear map  $\gamma : \mathbb{R}^{\binom{n-1}{2}} \rightarrow \mathbb{R}^d$  which maps correlation matrices to cut vectors:  $\gamma(\mathbf{x}\mathbf{x}^t) = \delta(S_{\mathbf{x}})$ , and thus  $\gamma(\text{COR}(n-1)) = \text{CUT}(n)$ . An inverse map is obtained by taking

$$\begin{aligned} x_{ii} &:= \delta_{in} \\ \text{and } x_{ij} = x_{ji} &:= \frac{1}{2}(x_{ii} + x_{jj} - \delta_{ij}) = \frac{1}{2}(\delta_{in} + \delta_{jn} - \delta_{ij}). \end{aligned}$$

The image of this inverse map consists of only symmetric matrices in  $\mathbb{R}^{\binom{n-1}{2}}$ , which describes the  $\binom{n}{2}$ -dimensional subspace of  $\mathbb{R}^{\binom{n-1}{2}}$  that is spanned by  $\text{COR}(n-1)$ .  $\square$

Note that the isomorphism of Lemma 21 is *not* a 0/1-equivalence — in fact the polytopes are not 0/1-equivalent, even in their full-dimensional versions. For example cut polytopes are centered (Lemma 19), but the correlation polytopes are not:  $\text{COR}(n)$  contains the point  $\frac{1}{2}\mathbf{1} = \frac{1}{2}(\mathbf{0} + \mathbf{1})$ , but this point lies in the boundary, since  $x_{11} \geq x_{12}$  is valid for all vertices of  $\text{COR}(n)$ , and not for all of them with equality.

We now record a remarkable property of the correlation polytopes (and of cut polytopes, via Lemma 21):

**Proposition 22**

Any three vertices of  $\text{COR}(n)$  determine a triangular face  $F \cong \Delta_2$ , that is,  $\text{COR}(n)$  is 3-neighborly.

**Proof.** Using the symmetry of  $\text{CUT}(n+1)$ , and its affine equivalence with  $\text{COR}(n)$ , we may assume that one of the three vertices of  $\text{COR}(n)$  is  $\mathbf{0}\mathbf{0}^t$ , while the others are  $\mathbf{u}\mathbf{u}^t$  and  $\mathbf{v}\mathbf{v}^t$ . The vectors  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$  span a 2-dimensional subspace  $U \subseteq \mathbb{R}^n$ , which may or may

not contain a fourth 0/1-vector  $\mathbf{y} \in \mathbb{R}^n$ , but no fifth vector. However, if there is such a fourth vector  $\mathbf{y}$ , then we may assume that  $\mathbf{y} = \mathbf{u} + \mathbf{v}$  (possibly after exchanging  $\mathbf{y}$  with  $\mathbf{u}$  or with  $\mathbf{v}$ ).

Now take a *generic* vector  $\mathbf{h} \in \mathbb{R}^n$  that is orthogonal to  $U$  — such a vector will satisfy  $\mathbf{h}^t \mathbf{u} = \mathbf{h}^t \mathbf{v} = \mathbf{h}^t \mathbf{0} = 0$ , and also  $\mathbf{h}^t \mathbf{y} = 0$  if  $\mathbf{y}$  exists, but  $\mathbf{h}^t \mathbf{x} \neq 0$  for any other  $\mathbf{x} \in \{0, 1\}^n$ . Then a little computation shows that the standard scalar product on  $\mathbb{R}^{n^2}$  with  $\mathbf{h}\mathbf{h}^t$  defines a *linear* function on  $\text{COR}(n)$  that is minimized by  $\mathbf{00}^t, \mathbf{uu}^t, \mathbf{vv}^t$ , and by  $\mathbf{yy}^t$  if this  $\mathbf{y}$  exists, but by no other vertex of  $\text{COR}(n)$ :

$$\begin{aligned} \langle \mathbf{h}\mathbf{h}^t, \mathbf{x}\mathbf{x}^t \rangle &= \sum_{1 \leq i, j \leq n} (\mathbf{h}\mathbf{h}^t)_{ij} (\mathbf{x}\mathbf{x}^t)_{ij} = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} h_i h_j x_i x_j = \\ &= \left( \sum_{1 \leq i \leq n} h_i x_i \right) \left( \sum_{1 \leq j \leq n} h_j x_j \right) = (\mathbf{h}^t \mathbf{x})^2 \geq 0. \end{aligned}$$

Now if there is no “fourth man”  $\mathbf{y}$ , then this proves that  $\text{conv}(\{\mathbf{uu}^t, \mathbf{vv}^t, \mathbf{00}^t\})$  is a (triangular) face of  $\text{COR}(n)$ . If, however,  $\mathbf{y} = \mathbf{u} + \mathbf{v}$  is present (that is,  $\mathbf{u} + \mathbf{v} \in \{0, 1\}^n$ , and thus  $\mathbf{u}^t \mathbf{v} = 0$ ), then we obtain that

$$F := \text{conv}(\{\mathbf{00}^t, \mathbf{uu}^t, \mathbf{vv}^t, (\mathbf{u} + \mathbf{v})(\mathbf{u} + \mathbf{v})^t\})$$

is a face of  $\text{COR}(n)$ . We have to show that this face is a tetrahedron, not a 2-face.

Since  $\mathbf{u}^t \mathbf{v} = 0$  with  $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$ , we can take indices  $i, j$  with  $u_i = 1, v_i = 0$  and  $v_j = 1, u_j = 0$ , so that

$$(\mathbf{uu}^t)_{ij} = u_i u_j = 0, \quad (\mathbf{vv}^t)_{ij} = v_i v_j = 0, \quad (\mathbf{yy}^t)_{ij} = (u_i + v_i)(u_j + v_j) = 1.$$

Thus  $\mathbf{yy}^t$  cannot be linearly dependent of  $\mathbf{uu}^t$  and  $\mathbf{vv}^t$ , and it is also clear that  $\mathbf{uu}^t$  and  $\mathbf{vv}^t$  are distinct 0/1-vectors and hence linearly independent. Thus  $\mathbf{uu}^t, \mathbf{vv}^t$  and  $(\mathbf{u} + \mathbf{v})(\mathbf{u} + \mathbf{v})^t$  are linearly independent, and hence  $F$  is a tetrahedron face of  $\text{COR}(n)$ .  $\square$

This result is best possible, since  $\text{CUT}(n)$  is not 4-neighborly in general: for this we note (for  $n \geq 3$ ) that

$$\delta(\emptyset) + \delta(\{1, 2\}) + \delta(\{1, 3\}) + \delta(\{2, 3\}) = \delta(\{1\}) + \delta(\{2\}) + \delta(\{3\}) + \delta(\{1, 2, 3\})$$

which implies that the four vectors on either side of the equation (which are distinct vectors for  $n \geq 4$ ) do *not* form a tetrahedron face of  $\text{CUT}(n)$ .

Proposition 22 implies that  $\text{CUT}(n)$  is *5-simplicial*, that is, all the 5-dimensional faces of  $\text{CUT}(n)$  are simplices (Exercise 13). On the other hand, the cut polytopes are not 6-simplicial:  $\text{CUT}(4)$  is 6-dimensional, but it is not a simplex. (Check **SIMPLICIALITY** for the cut polytopes in the **polymake** database!

### Corollary 23

For every dimension  $d = \binom{n}{2}$ , there is a 3-neighborly 0/1-polytope with more than  $2^{\sqrt{2d}-1/2}$  vertices.

**Proof.** Take CUT( $n$ ), whose number of vertices is  $2^{n-1}$ , with  $n = \frac{1}{2} + \sqrt{2d + \frac{1}{4}}$ .  $\square$

## 4.3 Many facets?

Here I would also like to give — at least — a rough estimate of the number of facets of CUT( $n$ ) for large  $n$ , but that seems not that easy to get. We note that

$$\text{CUT}(2) \cong \Delta_1 \quad \text{and} \quad \text{CUT}(3) \cong \Delta_3$$

are simplices, while computation in “small” dimensions (see the `polymake` database and SMAPO [15]) yields

- CUT(4) has dimension  $d = 6$  and  $16 = 1.5874^d$  facets,
- CUT(5) has dimension  $d = 10$  and  $56 = 1.4956^d$  facets,
- CUT(6) has dimension  $d = 15$  and  $368 = 1.4827^d$  facets,
- CUT(7) has dimension  $d = 21$  and  $116764 = 1.7430^d$  facets,
- CUT(8) has dimension  $d = 28$  and  $217093472 = 1.9849^d$  facets,
- CUT(9) has dimension  $d = 36$  and at least  $12246651158320 = 2.3097^d$  facets.

This suggests that CUT( $n$ ) has more than  $d^{cd}$  facets, for some  $c > 0$ : prove this!

## 5 The Size of Coefficients

Grötschel, Lovász & Schrijver [30], in their study of the ellipsoid method and its (fundamental) role in optimization, introduced the notion of the *facet complexity* of a polyhedron. This is roughly the maximal number of bits that is necessary to represent one single facet by an inequality (with rational coefficients). They showed that for polyhedra with bounded facet complexity, optimization and separation are equivalent. Thus, the complexity of the facets is more important in this context than the number of facets. The following will imply that the facet complexity of an  $n$ -dimensional 0/1-polytope is  $O(n^2 \log n)$ : this is a polynomial bound, and thus “good enough” for the ellipsoid method.

The question about the maximal facet complexity of 0/1-polytopes can also be phrased differently: it asks *How large integers (rationals) may occur in the  $\mathcal{H}$ -presentation of a 0/1-polytope?* The bad news is that the integer coefficients that appear in the inequality description of a 0/1-polytope may be *huge*. This is “bad”: it means that all kinds of algorithms, from cutting plane procedures to convex hull algorithms — used to compute the facets of a given polytope — are threatened by “integer overflow” even in the case of 0/1-polytopes.

The main source for this lecture is a recent paper by Noga Alon and Van H. Vu [4], which rests on a construction of Johan Hastad [32] from 1992.

## 5.1 Experimental evidence

What do we mean by the size of the coefficients of the facets? For this we write each facet-defining inequality of a full-dimensional (!) 0/1-polytope uniquely in the normal form

$$\pm c_0 \pm c_1 x_1 \pm c_2 x_2 \pm \dots \pm c_d x_d \geq 0,$$

for non-negative integers  $c_0, c_1, \dots, c_d$  with greatest common divisor 1. By the *greatest coefficient* we mean  $\max\{c_1, \dots, c_d\}$ . For example, the inequality

$$19 - 12x_1 - 18x_2 - 3x_3 - 1x_4 + 10x_4 - 11x_6 + 4x_7 - 5x_8 \geq 0$$

from `CF:8-9.poly` has greatest coefficient 18.

The concept of greatest coefficient is invariant under permuting coordinates (obviously), but also under switching (the substitution  $x_i \leftrightarrow 1 - x_i$  just switches the sign in front of  $c_i x_i$ , but not the size of the coefficient). It also changes the constant coefficient  $c_0$ , but we ignore these anyway. Note that for 0/1-polytopes we always have  $c_0 \leq c_1 + \dots + c_d$ , since the facet-defining inequality must be satisfied by some 0/1-point with equality. We will, however, apply the concept of “greatest coefficients” only in the case of full-dimensional polytopes, since otherwise the “defining inequality of a facet” is not unique, which makes things more complicated.

With these precautions, we can look up the largest coefficient  $\text{coeff}(d)$  that appears in a facet-defining inequality for a  $d$ -dimensional 0/1-polytope, and for low dimensions  $d$  we find the following:

$d$	$\text{coeff}(d)$	example
3	= 1	
4	= 2	<code>CF:4-5.poly</code>
5	= 3	<code>CF:5-6.poly</code>
6	= 5	<code>CF:6-7.poly</code>
7	= 9	<code>CF:7-8.poly</code>
8	= 18	<code>CF:8-9.poly</code>
9	$\geq 42$	<code>CF:9-10.poly</code>
10	$\geq 96$	<code>CF:10-11.poly</code>

Here the values for  $d \leq 8$  are from complete enumeration, the values for  $d > 8$  were taken from Aichholzer [1, p. 111]. The data for  $d \leq 10$  do not, however, provide enough evidence to guess the truth.

## 5.2 The Alon-Vũ theorem and some applications

Let  $A$  be a 0/1-matrix of size  $n \times n$ . The question *How bad can  $A$  be?* has many aspects. Here we will first look (again) at the maximal size of a determinant  $\det(A)$ . Then we get to the Alon-Vũ theorem about the maximal size of entries of  $A^{-1}$ , and to its consequences for the arithmetics (large coefficients) and the geometry (e. g. flatness) of 0/1-polytopes.

Denote by  $\rho_n$  the maximal determinant of a 0/1-matrix of size  $n \times n$ . The exact value of  $\rho_n$  seems to be known for all  $n < 18$ , except for  $n = 14$ , where the following table quotes a conjecture.

$n$	$\rho_n$
1	1
2	1
3	2
4	3
5	5
6	9
7	32
8	56
9	144
10	320
11	1458
12	3645
13	9477
14	25515    (? , Smith [54], Cohn [16])
15	131072
16	327680    ...

Matrices that achieve these values may be obtained from a web page by Dowdeswell, Neubauer, Solomon & Tumer [21].

### Lemma 24 (The Hadamard bound)

*The maximal determinant of a 0/1-matrix of size  $n \times n$  is bounded by*

$$\rho_n \leq 2 \left( \frac{\sqrt{n+1}}{2} \right)^{n+1}.$$

**Proof.** The Hadamard inequality states that the determinant of a square matrix is at most the product of the lengths of its columns, with equality (in the nonsingular case) if and only if all columns are orthogonal to each other. Applied to the case of a  $\pm 1$ -matrix  $\widehat{A}$  of size  $(n+1) \times (n+1)$ , this yields

$$\det(\widehat{A}) \leq \sqrt{n+1}^{n+1}. \tag{*}$$

We transfer this result to  $n \times n$  0/1-matrices  $A$  via Proposition 11, and get

$$\det(A) \leq \frac{\sqrt{n+1}^{n+1}}{2^n},$$

as claimed.  $\square$

A matrix  $\widehat{A} \in \{-1, +1\}^{(n+1) \times (n+1)}$  that achieves equality in (\*) is known as a *Hadamard matrix*. It is not hard to show that for this a condition is that  $n+1$  is 1, 2 or a multiple of 4. It is conjectured that these conditions are also sufficient, but for many values  $n+1 \geq 428$  this is not known. We refer to Hudelson, Klee & Larman [35] for an extensive, recent survey with pointers to the vast literature related to the Hadamard determinant problem. For the cases where  $n+1$  is not a multiple of 4 one has slightly better estimates (by a constant factor) than the Hadamard bound; see Neubauer and Radcliffe [49]. Certainly for our purposes we may consider the Hadamard bound as “essentially sharp.”

Now assume additionally that  $A \in \{0, 1\}^{n \times n}$  is invertible (of determinant  $\det(A) \neq 0$ ), consider the inverse  $B := A^{-1}$ , and let

$$\chi(A) := \max_{1 \leq i, j \leq n} |b_{ij}| = \|B\|_\infty$$

the largest absolute value of an entry of  $A^{-1}$ . These entries are — by Cramer’s rule — given by

$$b_{ij} = (-1)^{i+j} \det(A_{ij}) / \det(A),$$

where  $A_{ij}$  is obtained from  $A$  by deleting the  $i$ -th row and the  $j$ -th column. Let  $\chi(n)$  denote the maximal entry in the inverse of any invertible 0/1-matrix of size  $n \times n$ .

**Theorem 25 (Alon & Vñ [4])**

*The maximal absolute value of an entry in the inverse of an invertible 0/1-matrix of size  $n \times n$  can be bounded by*

$$\frac{n^{n/2}}{2^{2n+o(n)}} \leq \chi(n) \leq \rho_{n-1} \leq \frac{n^{n/2}}{2^{n-1}}.$$

*Furthermore, 0/1-matrices that realize the lower bound can be effectively constructed. (An even better lower bound, by a factor of  $2^n$ , is achieved in the case where  $n$  is a power of 2.)*

Before we look at the proof of this theorem, we derive two (quite immediate) applications to the geometry of 0/1-polytopes. First, let as above  $\text{coeff}(d)$  denote the largest  $c_i$  that can appear in a reduced inequality

$$\pm c_0 \pm c_1 x_1 \pm c_2 x_2 \pm \dots \pm c_d x_d \geq 0,$$

that defines a facet of a  $d$ -dimensional 0/1-polytope in  $\mathbb{R}^d$ . (Here the  $c_i$  are non-negative integers, with  $\text{gcd}(c_1, \dots, c_d) = 1$ ; by switching, we may assume that  $c_0 = 0$  if we want to.)

**Corollary 26 (Huge coefficients [4])**

The largest integer coefficient  $\text{coeff}(d)$  in the facet description of a full-dimensional 0/1-polytope in  $\mathbb{R}^d$  satisfies

$$\frac{(d-1)^{(d-1)/2}}{2^{2d+o(d)}} \leq \chi(d-1) \leq \text{coeff}(d) \leq \rho_{d-1} \leq \frac{d^{d/2}}{2^{d-1}}.$$

**Proof.** Let  $\{\mathbf{0}, \mathbf{v}_1, \dots, \mathbf{v}_{d-1}\} \subseteq \{0, 1\}^d$  be points that span a hyperplane  $H$  in  $\mathbb{R}^d$ , and let  $V = (\mathbf{v}_1, \dots, \mathbf{v}_{d-1})^t \in \{0, 1\}^{(d-1) \times d}$ . Then an equation that defines  $H$  is given by  $\mathbf{c}^t \mathbf{x} = 0$ , with  $c_i = \pm \det(V_i)$ , where  $V_i \in \{0, 1\}^{(d-1) \times (d-1)}$  is obtained from  $V$  by deleting the  $i$ th column. Thus we get the upper bound  $\text{coeff}(d) \leq \rho_{d-1}$  by definition.

For the lower bound  $\chi(d-1) \leq \text{coeff}(d)$  we start with a matrix  $A \in \{0, 1\}^{(d-1) \times (d-1)}$  such that  $\chi(A) = |\det A_{11}| / \det A = \chi(d-1)$ , and let  $V := (A, \mathbf{e}_1) \in \{0, 1\}^{(d-1) \times d}$ . Then  $|\det V_d| = |\det A|$ , while  $|\det V_1| = |\det A_{11}|$ . Thus for the coefficients  $c_i = \pm \det(V_i)$  of a corresponding inequality  $\mathbf{c}^t \mathbf{x} \geq 0$  we get

$$|c_1/c_d| = |\det A_{11}| / \det A = \chi(d-1),$$

and thus for any integral inequality which defines a facet that lies in our hyperplane  $H = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{c}^t \mathbf{x} = 0\}$  we have  $c_1 \geq \chi(d-1)$ .

A simplex for which this  $H$  defines a facet is, for example, given by the convex hull of  $\mathbf{0}$  and  $\mathbf{e}_1$  together with the rows of  $V$ . This simplex has determinant  $\det(A_{11})$ , which will be huge for the matrices  $A$  constructed for the Alon-Vũ theorem.  $\square$

**Corollary 27 (Flat 0/1-simplices [4])**

The minimal positive distance  $\text{flat}(d)$  of a 0/1-vector from a hyperplane that is spanned by 0/1-vectors in  $\mathbb{R}^d$  satisfies

$$\frac{2^{d-1}}{\sqrt{d}^{d+1}} \leq \frac{1}{\sqrt{d} \rho_{d-1}} \leq \text{flat}(d) \leq \frac{1}{\chi(d)} \leq \left(\frac{1}{d}\right)^{d/2} 2^{d(2+o(1))}.$$

**Proof.** Let  $H = \text{aff}\{\mathbf{0}, \mathbf{v}_2, \dots, \mathbf{v}_d\}$  be a hyperplane under consideration (we may assume that it contains the origin) and let  $\mathbf{v}_1 \in \{0, 1\}^d \setminus H$ . Then there is an integral normal vector  $\mathbf{c}$  to  $H$  with  $c_i = \pm \det(A_{i1})$ , for the square matrix  $A := (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d) \in \{0, 1\}^{d \times d}$ . From  $\mathbf{v}_1 \notin H$  we get  $|\mathbf{v}_1^t \mathbf{c}| \geq 1$ , while the length of  $\mathbf{c}$  is bounded by

$$\|\mathbf{c}\| \leq \sqrt{d} \|\mathbf{c}\|_\infty \leq \sqrt{d} \rho_{d-1},$$

and thus

$$\text{dist}(\mathbf{v}_1, H) = \frac{|\mathbf{v}_1^t \mathbf{c}|}{\|\mathbf{c}\|} \geq \frac{1}{\|\mathbf{c}\|} \geq \frac{1}{\sqrt{d} \rho_{d-1}}.$$

For the upper bound, take an  $A$  that achieves  $\chi(A) = |\det(A_{11})| / \det(A) = \chi(d)$ . Then

$$\frac{1}{\chi(d)} = \frac{|\det(A)|}{|\det(A_{11})|} = \frac{\text{Vol}(\text{conv}\{\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d\})}{\text{Vol}(\text{conv}\{\mathbf{0}, \mathbf{e}_1, \mathbf{v}_2, \dots, \mathbf{v}_d\})} = \frac{\text{dist}(\mathbf{v}_1, H)}{\text{dist}(\mathbf{e}_1, H)} \geq \frac{\text{dist}(\mathbf{v}_1, H)}{1},$$



where the last “=” is since we are considering two simplices with a common facet, and the inequality is from  $\text{dist}(\mathbf{e}_1, H) \leq \text{dist}(\mathbf{e}_1, \mathbf{0}) = 1$ .  $\square$

**Proof.** We now survey the main parts of the proof of the Alon-Vũ theorem, following [4].

**(1) The upper bound.** For the upper bound  $\chi(n) \leq \rho_{n-1}$  we use that the entries of  $A^{-1}$  can be written as

$$b_{ij} = (-1)^{i+j} \frac{\det(A_{ij})}{\det(A)},$$

where the cofactors  $A_{ij} \in \{0, 1\}^{(n-1) \times (n-1)}$  satisfy  $|\det(A_{ij})| \leq \rho_{n-1}$  by definition, and the invertible matrix  $A$  satisfies  $|\det(A)| \geq 1$  since it is integral.

**(2) Super-multiplicativity.** For the lower bound it is sufficient to construct “bad” matrices of size  $2^m \times 2^m$ , because of the following simple construction, which establishes

$$\chi(n_1 + n_2) \geq \chi(n_1) \cdot \chi(n_2).$$

Take “bad” invertible 0/1-matrices  $A$  and  $B$  of sizes  $n_1 \times n_1$  and  $n_2 \times n_2$ , such that  $\chi(A) = |\det A_{n_1, n_1} / \det A|$  and  $\chi(B) = |\det B_{11} / \det B|$ . Then the matrix

$$A \diamond B := \begin{pmatrix} A & & 0 \\ 0 & \cdots & 1 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{matrix} \\ \\ B \\ \end{matrix}$$

has determinant  $\det(A \diamond B) = \det(A) \cdot \det(B)$  and the submatrix

$$(A \diamond B)_{n_1, n_1+1} = \begin{pmatrix} & & * & & \\ & A_{n_1, n_1} & * & 0 & \\ 0 & \cdots & 1 & * & * \\ \vdots & & \vdots & B_{11} & \\ 0 & \cdots & 0 & & \end{pmatrix}$$

has determinant  $\det A_{n_1, n_1} \det B_{11}$ , which establishes

$$\chi(A \diamond B) \geq \chi(A) \chi(B).$$

Thus — modulo an annoying computation that you may find in [4, Sect. 2.4] — it suffices to establish the lower bound of the Alon-Vũ theorem for  $n = 2^m$ .

**(3) The construction.** Here comes the key part of the proof: an ingenious construction of a “bad”  $\pm 1$ -matrix whose size is a power of 2. Thus we prove that for  $n = 2^m$  one can construct an invertible matrix  $A \in \{+1, -1\}^{n \times n}$  with

$$\chi(A) = n^{n/2} \left(\frac{1}{2}\right)^{n+o(n)}$$

and then use Proposition 11. For this, the following is an explicit recipe. Perhaps you want to “do it” for  $m = 3$ ,  $n = 8$ ?

- (i) Choose an ordering  $\alpha_1, \alpha_2, \dots, \alpha_n$  on the collection of all  $2^m = n$  subsets of  $[m] = \{1, 2, \dots, m\}$ , such that  $|\alpha_i| \leq |\alpha_{i+1}|$  and  $|\alpha_i \Delta \alpha_{i+1}| \leq 2$  holds for all  $i$ . This is not hard to do.
- (ii) The matrix  $Q \in \{+1, -1\}^{n \times n}$  given by  $q_{ij} := (-1)^{|\alpha_i \cap \alpha_j|}$  is a symmetric Hadamard matrix (in fact, in lexicographic ordering of the rows and columns this is the “obvious” Hadamard matrix of order  $2^m$ ). Thus  $Q^2 = nI_n$ ,  $Q^{-1} = \frac{1}{n}Q$ , and  $\det(Q) = n^{n/2}$ .
- (iii) We construct a lower triangular matrix  $L \in \mathbb{Q}^{n \times n}$  row-by-row, with  $(1, 0, \dots, 0)$  as the first row. For  $i > 1$  define  $A_i := \alpha_{i-1} \cup \alpha_i$  and

$$F_i := \begin{cases} \{\alpha_s : \alpha_s \subseteq A_i, |\alpha_s \cap (\alpha_{i-1} \Delta \alpha_i)| = 1\} & \text{if } |\alpha_{i-1} \Delta \alpha_i| = 2, \\ \{\alpha_s : \alpha_s \subseteq A_i = \alpha_i\} & \text{if } |\alpha_{i-1} \Delta \alpha_i| = 1, \end{cases}$$

so that both  $\alpha_{i-1}, \alpha_i \in F_i$  and  $|F_i| = 2^k$  hold in both cases, for

$$k := |\alpha_i|.$$

Then for  $1 < i \leq n$  and  $1 \leq j \leq n$  we set

$$\ell_{ij} := \begin{cases} 0 & \text{if } \alpha_j \notin F_i, \\ \left(\frac{1}{2}\right)^{k-1} - 1 & \text{if } j = i - 1, \text{ and} \\ \left(\frac{1}{2}\right)^{k-1} & \text{otherwise.} \end{cases}$$

- (iv) We define  $A := LQ$ . A simple computation shows that  $a_{ij} \in \{+1, -1\}$  holds for all  $i, j$ . The determinant of  $A$  is  $2^{n-1}$ , since  $\det(Q) = n^{n/2} = 2^{m2^{m-1}}$  and

$$\det(L) = \prod_{i=1}^n \ell_{ii} = \prod_{k=1}^m \left(\frac{1}{2}\right)^{(k-1)\binom{m}{k}} = \left(\frac{1}{2}\right)^{\sum_{k=1}^m (k-1)\binom{m}{k}},$$

with

$$\sum_{k=1}^m (k-1)\binom{m}{k} = \sum_{k=1}^m k\binom{m}{k} - \sum_{k=1}^m \binom{m}{k} = m2^{m-1} - 2^m + 1.$$

Thus  $|\det(A)|$  has the minimal possible value for an invertible 0/1-matrix of size  $n \times n$ .

- (v) Take  $i_0 := 2 + m + \binom{m}{2}$ , which is the smallest index with  $|\alpha_{i_0}| \geq 3$ . We solve the system  $L\mathbf{x} = \mathbf{e}_{i_0}$ . This is easy since  $L$  is lower triangular:  $x_i = 0$  for  $i < i_0$ ,

$x_{i_0} = 1/\ell_{i_0 i_0} = 4$  since  $\ell_{i_0 i_0} = \frac{1}{2^{3-1}} = \frac{1}{4}$ ,  
and for  $i > i_0$  we can solve recursively:

$$x_i = (2^{k-1} - 1)x_{i-1} - \sum_{\alpha_j \in F_i \setminus \{\alpha_i, \alpha_{i-1}\}} x_j \quad \text{for } k = |\alpha_i|. \quad (*)$$

Using induction, we now verify that the  $x_i$  are positive and

$$x_i > (2^{k-1} - 2)x_{i-1} \quad \text{for } i > i_0. \quad (**)$$

Indeed, this holds for  $i = i_0 + 1$ , and by induction (with  $k \geq 3$ , so  $2^{k-1} - 2 \geq 2$ ) we have

$$x_{i-1} > 2x_{i-2} > 4x_{i-3} > \dots$$

Thus the sum in (\*) is smaller than

$$\frac{1}{2}x_{i-1} + \frac{1}{4}x_{i-1} + \dots = \sum_{t \geq 1} \frac{1}{2^t} x_{i-1} = x_{i-1}.$$

Using this estimate in (\*) we get for  $i > i_0$  that

$$x_i > (2^{k-1} - 1)x_{i-1} - x_{i-1} = (2^{k-1} - 2)x_{i-1}. \quad (***)$$

Iteration of the recursion (\*\*), with a start at  $x_{i_0} > 2$ , now yields

$$x_n > \prod_{k=3}^m (2^{k-1} - 2)^{\binom{m}{k}} = \prod_{k=3}^m 2^{(k-1)\binom{m}{k}} \prod_{k=3}^m \left(1 - \frac{2}{2^{k-1}}\right)^{\binom{m}{k}}$$

where the first product is  $2^N$  with

$$N = \sum_{k=1}^m (k-1) \binom{m}{k} - \binom{m}{2} = m2^{m-1} - 2^m - \binom{m}{2}$$

using the same sum as in (iv), and thus

$$2^N = 2^{m2^{m-1} - 2^m - \binom{m}{2}} = \frac{n^{n/2}}{2^{n + \binom{m}{2}}} = n^{n/2} \left(\frac{1}{2}\right)^{n + o(n)}.$$

Now we use that  $1 - x \geq \frac{1}{2^{2x}}$  for  $0 \leq x \leq \frac{1}{2}$  and thus estimate that the second product is at least  $(\frac{1}{2})^M$  for

$$M = 2 \sum_{k=3}^m \frac{1}{2^{k-2}} \binom{m}{k} < 8 \sum_{k=0}^m \frac{1}{2^k} \binom{m}{k} = 8 \left(\frac{3}{2}\right)^m = 8n^{\log 3/2} = o(n).$$

Taken together, we have verified that

$$x_n = n^{n/2} \left(\frac{1}{2}\right)^{n + o(n)}.$$

(vi) The rest is easy: to get the  $i_0$ -th column of  $A^{-1}$ , we solve the system

$$A\mathbf{y} = \mathbf{e}_{i_0} \iff LQ\mathbf{y} = \mathbf{e}_{i_0} \iff Q\mathbf{y} = \mathbf{x} \text{ and } L\mathbf{x} = \mathbf{e}_{i_0}.$$

But  $Q\mathbf{y} = \mathbf{x}$  is easy to solve because of  $Q^{-1} = \frac{1}{n}Q$ . Thus we obtain

$$B_{ii_0} = y_i = \frac{1}{n} \sum_{j=1}^n q_{ij}x_j.$$

Here  $|q_{ij}| = 1$  by construction and from (\*\*\*) , for  $k \geq 4$  ( $n \geq 16$ ), we have

$$x_n > 4x_{n-1} > 8x_{n-2} > \dots$$

which yields

$$B_{ii_0} = y_i > \frac{1}{n} \left( \frac{1}{2}x_n \right) = \frac{1}{2n}x_n \geq n^{n/2} \left( \frac{1}{2} \right)^{n+o(n)}.$$

Thus *all* entries of the  $i_0$ -column of  $A^{-1}$  are “huge.” □

### 5.3 More experimental evidence

The Alon-Vũ construction is completely explicit; you will find corresponding simplices (generated by Michael Joswig) as `MJ:16-17.poly` and as `MJ:32-33.poly` in the `polymake` database. The first one is a 16-dimensional simplex with “−451” appearing as a coefficient. The second one has dimension 32, and here you’ll find tons of coefficients like “4964768222” that are indeed large enough to cause trouble for any conventional single-precision arithmetic system . . .

## 6 Further Topics

There are so many interesting aspects of 0/1-polytopes, and so little time and space. In this section, I am therefore collecting brief notes about three further topics, together with pointers to the literature that I’d hope you’ll follow.

### 6.1 Graphs

General facts about graphs of polytopes apply in the 0/1-context, but there are new phenomena appearing — the most tantalizing perhaps being the Mihail-Vazirani conjecture. But we start with a basic fact that is true for all (bounded, convex) polytopes, and hence need not be proved in our more special context.

**Theorem 28 (Balinski [6]; Holt & Klee [34])**

- (1) *The graph of every  $d$ -dimensional polytope is vertex  $d$ -connected; that is, there are  $d$  vertex-disjoint paths between any pair of vertices.*
- (2) *For any generic linear objective function (such that no two vertices get the same value), there are  $d$  monotone vertex-disjoint paths from minimum to maximum.*

In a setting of general (convex, bounded) polytopes the first part of this, “Balinski’s Theorem,” is a classic. The second part is a rather recent strengthening observed by Holt & Klee [34]: it implies the first part since for any two distinct vertices of a polytope we may assume that they are the unique minimal and the unique maximal vertex for a generic linear function, *after a projective transformation* [58, p. 74]. One peculiar phenomenon is that this reduction does not work in a setting of 0/1-polytopes: projective transformations do not preserve 0/1-polytopes.

The second result for this section is an example of an important and still unsolved problem from the theory of general polytopes (see [58, Sect. 3.3]) which becomes quite trivial when specialized to 0/1-polytopes — as was first noticed by Denis Naddef.

**Theorem 29 (The Hirsch conjecture for 0/1-polytopes: Naddef [48])**

*The diameter of the graph of a  $d$ -dimensional 0/1-polytope  $P \subseteq \mathbb{R}^n$  is at most*

$$\text{diam}(G(P)) \leq d,$$

*with equality if and only if  $P$  is (affinely equivalent to) a  $d$ -dimensional 0/1-cube. In particular, this implies that*

$$\text{diam}(G(P)) \leq n - d,$$

*where  $n$  is the number of facets of  $P$ .*

**Proof.** We get the first inequality by induction on dimension, the case  $d = 1$  being trivial. If the two vertices in question lie in a common facet of  $[0, 1]^d$ , then we can restrict to the corresponding trivial face of  $P$  of dimension at most  $d - 1$ , and we are thus done by induction. Hence we may assume that  $\mathbf{v}$  and  $\mathbf{u}$  are opposite vertices of  $[0, 1]^d$ , and by symmetry only need to consider the case where  $\mathbf{v} = \mathbf{0}$  and  $\mathbf{u} = \mathbf{1}$ .

But the vertex  $\mathbf{u} = \mathbf{1}$  is connected to some neighboring vertex  $\mathbf{u}'$ , and this neighbor is contained in some trivial face  $P_i^0$ , whose diameter is at most  $d - 1$  by induction. Thus

$$d(\mathbf{v}, \mathbf{u}) \leq d(\mathbf{v}, \mathbf{u}') + d(\mathbf{u}', \mathbf{u}) \leq (d - 1) + 1 = d.$$

For the second statement, we may assume (using induction on dimension) that the two vertices in question do not lie on a common facet. Thus the polytope has at least  $n \geq 2d$  distinct facets.  $\square$

Our third item in this section is a conjecture that’s just plain wrong for general polytopes, but may be true in the 0/1-setting.

**Conjecture 30 (Mihail-Vazirani [25, Sect. 7])**

The graph of every 0/1-polytope is a good expander. Specifically, for every partition  $V = S \uplus \bar{S}$  of the vertex set, the polytope  $P(V)$  has at least

$$E(S, \bar{S}) \geq \min\{|S|, |\bar{S}|\}$$

edges between  $S$  and  $\bar{S}$ .

Remark: This may be very false. It does not seem to be trivial.

## 6.2 Triangulations

A very basic question is the following: *How many simplices are needed to triangulate the  $d$ -dimensional 0/1-cube?* Here the exact answer depends on the exact definitions: for example, let us assume that we want proper triangulations where all simplices are required to fit together face-to-face, and not only subdivisions, or (even worse) coverings. Let us also assume that we only admit triangulations without new vertices. (In general polytopes, new vertices *do* help — see Below et al. [8].)

In this setting, let  $\text{triang}(d)$  be the smallest number of simplices in a triangulation of  $[0, 1]^d$ . Then we can draw up a little table,

$d$	$\text{triang}(d)$
1	1
2	2
3	5
4	16
5	67
6	308
7	1493

combining many earlier results with those of Hughes [36, 37] and Hughes & Anderson [38, 39]. For  $d = 8$ , all we have seem to be the bounds  $5522 \leq \text{triang}(8) \leq 11944$ .

One of several curious effects in this context is that *not every*  $d$ -dimensional 0/1-polytope can be triangulated into at most  $\text{triang}(d)$  simplices: for example, for the 6-dimensional half-cube `HC:7-64.poly`, the convex hull of all 0/1-vectors of even weight, one knows that the minimal number of simplices in a triangulation is  $1756 > 1493 = \text{triang}(7)$  (Hughes & Anderson [39]).

A lower bound is certainly given by the maximal volume of a 0/1-simplex,

$$\text{triang}(d) \geq \frac{d!}{\rho_d},$$

but this bound is not very good. (For example, for  $d = 3$  it yields only  $\text{triang}(d) \geq 3$ ). However, it can be refined by giving greater weight to the simplices “near the boundary,”

which have lower volume, but are needed to fill the 0/1-cube. A very elegant and powerful version of such a lower bound was given by Smith [55] using hyperbolic geometry.

A good quantity to consider is

$$\sqrt[d]{\frac{\#\text{simplices}}{d!}},$$

called the *efficiency* of a triangulation. This number is at most 1 for any triangulation that uses no “extra vertices.” Haiman [33] showed that the limit

$$L := \lim_{d \rightarrow \infty} \sqrt[d]{\frac{\text{triang}(d)}{d!}}$$

exists, and that the efficiency of any example can also be achieved asymptotically, that is,

$$\sqrt[d]{\frac{\#\text{simplices}}{d!}} \leq L$$

holds for every triangulation without new vertices. The best upper bound on  $L$  up to now seems to be the one provided by Santos [52]:

$$L \leq \sqrt[3]{\frac{7}{12}} \approx 0.836.$$

One would, however, expect that the limit  $L$  is zero.

### 6.3 Chvátal-Gomory ranks

Interesting questions are related to the rounding procedures of integer programming that try to recover the convex hull  $P_I := \text{conv}(P \cap \mathbb{Z}^d)$  from an inequality description of a polytope  $P \subseteq [0, 1]^d$ .

In particular, Chvátal-Gomory rounding steps replace  $P$  by

$$P' := \bigcap_{H \supset P} H_I,$$

where the intersection is taken over all closed halfspaces  $H$  that contain  $P$ . The integer closure  $H_I$  of a halfspace  $H$  is easy to compute: make the left-hand side of the inequalities integral with greatest common divisor one, and then round the right-hand side. It was proved by Chvátal that a finite number of such closure operations lead from a bounded polytope  $P$  to its integer hull — *but how many steps are needed?* This quantity is known as the *Chvátal-Gomory rank* or *CG-rank* of the polytope  $P$ . We refer to the thorough treatment by Schrijver [53] for details and references.

Bockmayr, Eisenbrand, Hartmann & Schulz [11] noticed recently that for polytopes in the 0/1-cube,  $P \subseteq [0, 1]^d$  the Chvátal-Gomory rank is bounded by a polynomial in  $d$ . An

improvement of Eisenbrand & Schulz [23] establishes that for  $P \subseteq [0, 1]^d$  the CG-rank is bounded by

$$(1 + \varepsilon)d \leq \text{CGr}(d) \leq 3d^2 \log(d)$$

for some  $\varepsilon > 0$ .

But how about a good lower bound? Riedel [51] has implemented a procedure to compute the CG-rank for polytopes, and he has provided explicit, low-dimensional examples  $P \subseteq [0, 1]^d$  for which the CG-rank exceeds the dimension; so we know

$$\begin{aligned} \text{CGr}(3) &= 3 \\ \text{CGr}(4) &= 5 \\ \text{CGr}(5) &= 6 \text{ (?) } \\ \text{CGr}(6) &\geq 8 \\ \text{CGr}(7) &\geq 9 \end{aligned}$$

But can anyone provide a lower bound that is more than simply linear?

## 7 Problems and Exercises

1. Is it true that every simple 0/1-polytope is a product of simplices?  
(This question was answered by Kaibel & Wolff [41].)
2. Estimate the maximal vertex degree of a  $d$ -dimensional 0/1-polytope.  
(Hint: OA:5-18.poly)
3. Classify the 0/1-polytopes of diameter  $\sqrt{2}$ .
4. \*Bound the maximal number of vertices for a  $d$ -dimensional 2-neighborly 0/1-polytope. (Corollary 23 yields an exponential lower bound.)
5. Show that every 0/1-polytope without a triangle face is a  $d$ -cube.  
(Volker Kaibel noticed that this follows from a result of Blind & Blind [10].)
6. \*Is it true that a simplicial 0/1-polytope of dimension  $d$  has at most  $2^d$  facets?  
Is it true that every simplicial 0/1-polytope of dimension  $d$  with  $2d$  vertices is centrally symmetric and thus is a cross polytope with exactly  $2^d$  vertices?  
(This is true for  $d \leq 6$ , according to Aichholzer's enumerations.)
7. Estimate the probability that the determinant of a random  $(n \times n)$ -matrix with entries in  $\mathbb{Z}_2$  vanishes (for large  $n$ ). Compare your result with that claimed in [46].
8. Show that for every fixed  $\varepsilon > 0$ , all the trivial faces of a random 0/1-polytope with  $(2 - \varepsilon)d$  vertices are simplices, with probability tending to 1 for  $d \rightarrow \infty$ .  
(Volker Kaibel)



9. Prove the Szekeres-Turán theorem: The expected value of the determinant  $\det(C)$  of a random  $\pm 1$ -matrix  $C \in \{-1, +1\}^{n \times n}$  is zero, but the expected value of the squared determinant is exactly  $n!$ :

$$E(\det(C)^2) = n!.$$

Hint, by Bernd Gärtner: Use  $\det(C) = \sum_{i=1}^n (-1)^{i-1} c_{1i} \det(C_{1i})$ , and analyze the expected values of the summands in

$$\det(C)^2 = \sum_{i=1}^n (\det(C_{1i}))^2 + \sum_{i \neq j} (-1)^{i+j} c_{1i} c_{1j} \det(C_{1i}) \det(C_{1j}).$$

10. What is the largest absolute value of the determinant of an  $n \times n$  matrix with coefficients in  $\{-1, 0, 1\}$ ? With coefficients in the interval  $[0, 1]$ ? With coefficients in  $[-1, 1]$ ?  
(It is reported that this is a question that was asked by L. Collatz at an international conference in 1961, and answered a year later by Ehlich & Zeller [22]. Your answer should be in terms of  $\rho_n$  resp.  $\rho_{n-1}$ .)

11. Show that  $\text{CUT}(k)$  is (0/1-isomorphic to) a face of  $\text{CUT}(n)$ , for  $k \leq n$ .  
 12. Prove that  $[0, 1]$  is an edge of the correlation polytope  $\text{COR}(n)$ .  
 13. Show that  $\text{CUT}(n)$  is 6-simplicial: every 5-dimensional face is a simplex.  
 14. Show that the *metric polytope*

$$\text{MET}(n) := \left\{ X \in [0, 1]^d : \begin{array}{l} x_{ij} - x_{ik} - x_{jk} \leq 0 \quad \text{and} \\ x_{ij} + x_{ik} + x_{jk} \leq 2 \quad \text{for distinct } i, j, k \in [n] \end{array} \right\}$$

is an LP-relaxation of  $\text{CUT}(n)$ : it satisfies  $\text{CUT}(n) = \text{conv}(\text{MET}(n) \cap \mathbb{Z}^d)$ , where  $d = \binom{n}{2}$  is the dimension of  $\text{CUT}(n) \subseteq \text{MET}(n) \subseteq \mathbb{R}^d$ .

\*Estimate the CG-rank of  $\text{MET}(n)$ .

15. How do the inequalities for a 0/1-polytope transform into the inequalities for the corresponding  $(+1/-1)$ -polytope?  
 16. Give more and better examples of “large” coefficients appearing in the facet-defining inequalities of 0/1-polytopes.  
 17. Show that every triangulation of  $\Delta_k \times \Delta_\ell$  without new vertices has exactly  $\binom{k+\ell}{k}$  facets.  
 18. \*For which dimensions  $d > 1$  and integers  $k$  ( $1 \leq k \leq d$ ) does there exist a regular  $d$ -dimensional 0/1-simplex of edge length  $\sqrt{k}$ ?  
(Show that this is equivalent to the existence of a matrix  $M \in \{0, 1\}^{d \times d}$  with  $M^t M = \frac{k}{2}(I_d + \mathbf{1}^t \mathbf{1})$ , so in particular  $k$  must be even. Show that the case  $k = d$  is equivalent to the famous Hadamard determinant problem.)

19. \*For which  $d$  is there a regular  $d$ -dimensional 0/1-cross polytope?
20. For which  $E \subseteq \binom{[n]}{2}$  is  $P(E) = \text{conv}\{\mathbf{e}_i + \mathbf{e}_j : \{i, j\} \in E\}$  a simplex?  
 Show that every such simplex of dimension  $d = \binom{n}{2}$  has normalized volume  $\frac{2^k}{d!}$  for some  $k \geq 0$ . [17]
21. Estimate the volumes of the Birkhoff polytopes

$$B_{n+1} := \{X \in [0, 1]^{n \times n} : \mathbf{1}^t X \leq \mathbf{1}^t, X \mathbf{1} \leq \mathbf{1}, \mathbf{1}^t X \mathbf{1} \geq n - 1\}.$$

(see BIR3:4-6.poly, BIR4:9-24.poly, ...). The exact value of the volume of  $B_{n+1}$ , which is some integer divided by  $n^2!$ , is known for  $n \leq 7$ , due to Chan & Robbins [13].

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