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Bicriteria Traffic Assignment

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Abstract. The classical traffic assignment problem determines the equilibrium flows on a congested network with many infinitesimal users who choose paths to minimize their own travel times. Travelers care about cost (including tolls) as well as time, and different travelers put different weights on these two criteria. We study a bicriteria traffic assignment problem in which travelers differ in their tradeoff between a flow-independent cost, such as tolls, and a flow-dependent cost, such as travel time. Existing formulations require either finitely many user classes or positive preference densities, including formulations for positive preference densities that are either infinite-dimensional or nonconvex. We formulate a finite-dimensional convex optimization problem for arbitrary preference distributions with bounded support, including distributions with a mixture of atoms, densities, and gaps in their support, and we show how to convert an optimal solution to an equilibrium of the bicriteria traffic assignment problem. We propose an algorithm that alternates between solving a convex problem for a (typically small) number of paths, and solving a parametric shortest-path problem. Computational experiments on benchmark networks illustrate the performance of the algorithm relative to existing methods.

1. Bicriteria Traffic Assignment Problem

Individual users, who want to travel from their origins to their destinations in a network, choose their preferred paths (we use the term *user* instead of traveler/decision maker/player/agent). An important problem is the determination of Wardrop equilibrium flows in the network, in which no user wants to change their chosen path. We consider a problem with many infinitesimal (nonatomic) users, so that an individual user's path choice does not affect the desirability of the paths, that is, each individual user can regard the desirability of the paths as independent of the user's own path choice. In addition, different users have different path preferences. For example, one type of user may prefer a path from origin to destination with the least travel time, and another type of user may prefer a path from the same origin to the same destination with the least monetary travel cost (such as transit fees or tolls). We consider the bicriteria traffic assignment problem (BiTAP) in which users choose paths from their origins to their destinations that minimize a weighted combination

of two criteria, one called (travel) time and the other called cost or toll. The travel time along a path depends on the flows on each of the arcs of the path, typically due to traffic congestion. As pointed out in Marcotte et al. (1996) and Marcotte and Zhu (1997), a necessary condition for the equilibrium conditions to coincide with the first-order optimality conditions of an optimization problem, is that one of the criteria does not depend on flow rates. Therefore, in the standard BiTAP, the cost/toll along a path does not depend on the flows on the path. Different users use different weights for the two criteria, and users are partitioned into types according to their relative weights. If the number of user types is finite, that is, the user type distribution has finite support, then a modification, with additional user type indices, of the various formulations for the single criterion traffic assignment problem can be used (see, e.g., Dafermos 1972). If the number of user types is infinite, then alternative approaches are needed.

Next we set notation that will be used throughout, and then we present various formulations of the BiTAP. Let $(\mathcal{N}, \mathcal{A})$ denote the finite directed network. For each origin-destination (OD) pair $(u, v) \in \mathcal{N}^2$, let $D^{(u,v)} \geq 0$ denote the demand rate (flow rate per unit time) for flow from u to v, and let $\Pi := \{(u, v) \in \mathcal{N}^2 : D^{(u,v)} > 0\}$ denote the OD pairs with positive demand. We use (u, v) to denote an OD pair when it is necessary to be specific about the origin or destination, otherwise we use π. Unless noted otherwise, it is assumed that D^π is input; this is called the fixed demand case. For each OD pair $\pi \in \Pi$, let $\mathcal{R}^\pi$ denote the set of paths in the network for OD pair π, for each arc $a \in \mathcal{A}$, let $\mathcal{R}^\pi_a$ denote the set of paths for OD pair π that use arc a, and for each $r \in \mathcal{R}^\pi$, let $\mathcal{A}^\pi_r$ denote the set of arcs on path r. For each arc $a \in \mathcal{A}$, let $s_a \in \mathbb{R}_+$ denote the cost per unit flow on arc a that does not depend on the flow rates, and let $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ denote the travel time on arc a as a function of the total flow rate on arc a. We assume that t_a is nondecreasing. A user with weight $w \in \mathbb{R}_+$ for the cost relative to the travel time has preference consistent with bicriteria cost $c_a(w, x_a) := t_a(x_a) + w s_a$ when x_a is the total flow rate on arc a. Let $x := (x_a, a \in \mathcal{A})$. It is assumed that the bicriteria cost $C^\pi_r(w, x)$ per unit flow for a path $r \in \mathcal{R}^\pi$ is the sum of the arc costs per unit flow, $C^\pi_r(w, x) = \sum_{a \in \mathcal{A}^\pi_r} c_a(w, x_a)$. For each OD pair $\pi \in \Pi$, and each path $r \in \mathcal{R}^\pi$, let $S^\pi_r := \sum_{a \in \mathcal{A}^\pi_r} s_a$ be the sum of the arc costs per unit flow, and let $T^\pi_r(x) := \sum_{a \in \mathcal{A}^\pi_r} t_a(x_a)$ be the total travel time as a function of arc flows x. For each OD pair $\pi \in \Pi$, and each path $r \in \mathcal{R}^\pi$, let f^π_r denote the flow on path r, and let $f := (f^\pi_r, \pi \in \Pi, r \in \mathcal{R}^\pi)$. Given path flows f, the flow x_a on each arc a is uniquely given by $x_a = \sum_{\pi \in \Pi} \sum_{r \in \mathcal{R}^\pi_a} f^\pi_r$. However, given arc flows x, the path flows f are not unique in general.

As mentioned above, different users have different weights; let G^π denote the cumulative distribution function of weight w among travelers for OD pair $\pi \in \Pi$, and let $[\underline{w}^\pi, \overline{w}^\pi] \subset \mathbb{R}_+$ denote the smallest closed interval that contains the support of G^π.

For each OD pair $\pi \in \Pi$, we choose a total order $\prec$ on the set of paths $\mathcal{R}^\pi$ such that for any $r, r' \in \mathcal{R}^\pi$ it holds that if $r \prec r'$ (implying $r \neq r'$) then $S^\pi_r \geq S^\pi_{r'}$. (Unlike previous papers on the BiTAP, we do not require the order to be unique, that is, we do not require that if $r \prec r'$ then $S^\pi_r > S^\pi_{r'}$.) The notation $\prec$ does not show the dependence of the order on π, because it will be clear from the context.

Given arc flows x, a path $r \in \mathcal{R}^\pi$ is called efficient if there is an interval $[\underline{w}^\pi_r, \overline{w}^\pi_r] \subset [\underline{w}^\pi, \overline{w}^\pi]$ with $\underline{w}^\pi_r \leq \overline{w}^\pi_r$ such that $C^\pi_r(w,x) \leq C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$ and $w \in [\underline{w}^\pi_r, \overline{w}^\pi_r]$, and a path $r \in \mathcal{R}^\pi$ is called strictly efficient if there is an interval $[\underline{w}^\pi_r, \overline{w}^\pi_r] \subset [\underline{w}^\pi, \overline{w}^\pi]$ with $\underline{w}^\pi_r < \overline{w}^\pi_r$ such that $C^\pi_r(w,x) < C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi \setminus \{r\}$ and $w \in (\underline{w}^\pi_r, \overline{w}^\pi_r)$. Next we point out some basic properties that have been useful in previous research on the BiTAP. Consider two efficient paths $r, r' \in \mathcal{R}^\pi$, and suppose that $r \prec r'$ (thus $S^\pi_r \geq S^\pi_{r'}$). Consider any $w \in [\underline{w}^\pi_r, \overline{w}^\pi_r]$ and $w' \in [\underline{w}^\pi_{r'}, \overline{w}^\pi_{r'}]$. Then $T^\pi_r(x) + wS^\pi_r \leq T^\pi_{r'}(x) + wS^\pi_{r'}$ and thus $T^\pi_{r'}(x) - T^\pi_r(x) \geq w\left(S^\pi_r - S^\pi_{r'}\right) \geq 0$, hence $T^\pi_{r'}(x) \geq T^\pi_r(x)$. Similarly, $T^\pi_{r'}(x) - T^\pi_r(x) \leq w'\left(S^\pi_r - S^\pi_{r'}\right)$. Previous papers assumed that different paths have different costs so that $S^\pi_r > S^\pi_{r'}$, in which case it can be concluded that $w \leq w'$. The assumption that $S^\pi_r > S^\pi_{r'}$ is awkward when these costs represent tolls (as in the canonical application), because in many applications there are many paths with no tolls. Although the path costs could be perturbed for the purpose of numerical work, we show that this is unnecessary, and more importantly, that the focus on weight intervals $[\underline{w}^\pi_r, \overline{w}^\pi_r]$ is unnecessarily restrictive without contributing to efficient computation. For the purpose of presentation and comparison with previous work, we will also identify weight intervals $[\underline{w}^\pi_r, \overline{w}^\pi_r]$, and we will show that in any case (even if $S^\pi_r = S^\pi_{r'}$), the intervals can be chosen such that $\overline{w}^\pi_r \leq \underline{w}^\pi_{r'}$. The choices of intervals $[\underline{w}^\pi_r, \overline{w}^\pi_r]$ and $[\underline{w}^\pi_{r'}, \overline{w}^\pi_{r'}]$ may not be unique; for example, if $r, r' \in \mathcal{R}^\pi$ are both efficient and $S^\pi_r = S^\pi_{r'}$ (and thus $T^\pi_r(x) = T^\pi_{r'}(x)$), then there is a common interval $[\underline{w}^\pi_{r,r'}, \overline{w}^\pi_{r,r'}] \subset [\underline{w}^\pi, \overline{w}^\pi]$ with $\underline{w}^\pi_{r,r'} \leq \overline{w}^\pi_{r,r'}$ such that $C^\pi_r(w,x) = C^\pi_{r'}(w,x) \leq C^\pi_{r''}(w,x)$ for all $r'' \in \mathcal{R}^\pi$ and $w \in [\underline{w}^\pi_{r,r'}, \overline{w}^\pi_{r,r'}]$. In such a case, if $r \prec r'$, then we will choose intervals $[\underline{w}^\pi_r, \overline{w}^\pi_r], [\underline{w}^\pi_{r'}, \overline{w}^\pi_{r'}] \subset [\underline{w}^\pi_{r,r'}, \overline{w}^\pi_{r,r'}]$ such that $\overline{w}^\pi_r \leq \underline{w}^\pi_{r'}$. Also, if paths $r, r' \in \mathcal{R}^\pi$ are both efficient, and $r \prec r'$, and there is no strictly efficient path $r'' \in \mathcal{R}^\pi$ such that $r \prec r'' \prec r'$, then the intervals $[\underline{w}^\pi_r, \overline{w}^\pi_r]$ and $[\underline{w}^\pi_{r'}, \overline{w}^\pi_{r'}]$ can be chosen such that $\overline{w}^\pi_r = \underline{w}^\pi_{r'}$. This allows the possibility that there is an efficient path $r'' \in \mathcal{R}^\pi$ such that $r \prec r'' \prec r'$ and $\overline{w}^\pi_r = \underline{w}^\pi_{r''} = \overline{w}^\pi_{r''} = \underline{w}^\pi_{r'}$. Thus, given arc flows x, there is a nonempty set $\{r^1, \ldots, r^k\} \subset \mathcal{R}^\pi$ of efficient paths and a corresponding

sequence of thresholds $\underline{w}^\pi = w^\pi_{r^0} \le w^\pi_{r^1} \le \cdots \le w^\pi_{r^{k-1}} \le w^\pi_{r^k} = \overline{w}^\pi$ such that for all $i = 1, \ldots, k$ it holds that $C^\pi_{r^i}(w,x) \le C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$ and $w \in [w^\pi_{r^{i-1}}, w^\pi_{r^i}]$. As mentioned above, the set $\{r^1, \ldots, r^k\} \subset \mathcal{R}^\pi$ and the thresholds $w^\pi_{r^1} \le \cdots \le w^\pi_{r^{k-1}}$ do not have to be unique, as illustrated in Example 1.

EXAMPLE 1. Fix an OD pair $\pi \in \Pi$ with support interval $[\underline{w}^\pi, \overline{w}^\pi] = [0, 0.14]$, arc flows x, and a path set $\mathcal{R}^\pi = \{r_1, r_2, \ldots, r_{10}\}$ indexed so that $r_1 \prec r_2 \prec \cdots \prec r_{10}$. Let $(S^\pi_{r_1}, S^\pi_{r_2}, \ldots, S^\pi_{r_{10}}) = (5, 4, 4, 3.7, 3.35, 3, 2.6, 2.25, 2, 1)$ and $(T^\pi_{r_1}(x), T^\pi_{r_2}(x), \ldots, T^\pi_{r_{10}}(x)) = (0.04, 0.05, 0.05, 0.135, 0.16, 0.12, 0.18, 0.235, 0.19, 0.31)$. Figure 1 shows the points $(S^\pi_{r_i}, T^\pi_{r_i}(x))$ for $i = 1, \ldots, 10$. As x changes, $T^\pi(x)$ changes, and thus the points move up or down. Note that, at the given x, the cost pairs $(S^\pi_{r_i}, T^\pi_{r_i}(x))$ for paths r_1, r_2, r_3, r_6, r_9, and r_{10} are on the efficient frontier, whereas the cost pairs $(S^\pi_{r_i}, T^\pi_{r_i}(x))$ for paths r_4, r_5, r_7, and r_8 are not on the efficient frontier.

Next we identify thresholds $\underline{w}^\pi = w^\pi_0 \le w^\pi_1 \le \cdots \le w^\pi_{k-1} \le w^\pi_k = \overline{w}^\pi$ such that for all efficient paths r_i it holds that $C^\pi_{r_i}(w,x) \le C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$ and $w \in [w^\pi_{i-1}, w^\pi_i]$. Note that $C^\pi_{r_1}(w,x) \le C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$ and $w \in [w^\pi_0, w^\pi_1] = [0, 0.01]$, and that the slope between $(S^\pi_{r_1}, T^\pi_{r_1}(x))$ and $(S^\pi_{r_2}, T^\pi_{r_2}(x))$ is (the negative of) 0.01. Next, $(S^\pi_{r_2}, T^\pi_{r_2}(x)) = (S^\pi_{r_3}, T^\pi_{r_3}(x))$, that is, paths r_2 and r_3 are exactly tied. Thus, $C^\pi_{r_2}(w,x) = C^\pi_{r_3}(w,x) \le C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$ and $w \in [0.01, 0.07]$, where the slope between $(S^\pi_{r_2}, T^\pi_{r_2}(x))$ and $(S^\pi_{r_6}, T^\pi_{r_6}(x))$ is (the negative of) 0.07. Because $r_2 \prec r_3$, their intervals may be chosen as $[w^\pi_1, w^\pi_2] = [0.01, \eta]$ and $[w^\pi_2, w^\pi_3] = [\eta, 0.07]$ for any $\eta \in [0.01, 0.07]$. Next, because paths r_4 and r_5 are inefficient, w^π_4 and w^π_5 have to be chosen as $w^\pi_4 = w^\pi_5 = w^\pi_3 = 0.07$. However, we have to keep in mind that paths r_4 and r_5 are inefficient, and thus it does not hold that $C^\pi_{r_4}(w,x) \le C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$ and $w \in [w^\pi_3, w^\pi_4] = [0.07, 0.07]$, and similarly for path r_5. Next, note that point $(S^\pi_{r_6}, T^\pi_{r_6}(x))$ lies on the line segment between points $(S^\pi_{r_3}, T^\pi_{r_3}(x))$ and $(S^\pi_{r_9}, T^\pi_{r_9}(x))$. Thus, w^π_6 also has to be chosen as $w^\pi_6 = w^\pi_5 = 0.07$. However, in this case path r_6 is efficient, and thus it holds that $C^\pi_{r_6}(w,x) \le C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$ and $w \in [w^\pi_5, w^\pi_6] = [0.07, 0.07]$. Thus, path r_6 is not strictly efficient. Nevertheless, if the distribution G^π has a point mass at w^π_6, then path r_6 may be preferred by a positive amount of demand, even though $[w^\pi_5, w^\pi_6]$ is a singleton. That does not mean that all the demand corresponding to the point mass at w^π_6 has to flow on path r_6 — the demand corresponding to the point mass at w^π_6 is partitioned among paths r_2, r_3, r_6, and r_9. In general, the demand corresponding to any point mass of G^π at w is partitioned among all the paths r_i such that $C^\pi_{r_i}(w,x) \le C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$. Also note that the status of path r_6 is sensitive to small perturbations of $T^\pi_{r_6}(x)$. For example, if x changes by a small amount to x' and as a result $T^\pi_{r_6}(x)$ increases to $T^\pi_{r_6}(x')$ that is slightly above the line segment between points $(S^\pi_{r_3}, T^\pi_{r_3}(x'))$ and

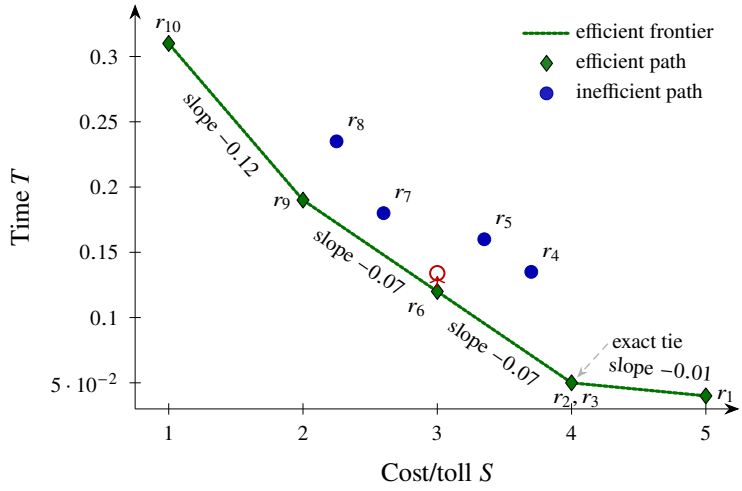


Figure 1 Bicriteria cost pairs $(S^\pi_{r_i}, T^\pi_{r_i}(x))$ for different paths r_i for a given x, and the corresponding efficient frontier, as in Example 1.

$(S^\pi_{r_9}, T^\pi_{r_9}(x'))$, as shown by the red circle in Figure 1, then path r_6 ceases to be efficient and the flow on path r_6 can jump from a positive amount to zero. Continuing in the same way, paths r_7 and r_8 are inefficient, and thus w^π_7 and w^π_8 have to be chosen as $w^\pi_7 = w^\pi_8 = w^\pi_6 = 0.07$. Next, note that $C^\pi_{r_9}(w,x) \le C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$ and $w \in [w^\pi_8, w^\pi_9] = [0.07, 0.12]$, and that the slope between $(S^\pi_{r_9}, T^\pi_{r_9}(x))$ and $(S^\pi_{r_{10}}, T^\pi_{r_{10}}(x))$ is (the negative of) 0.12. Finally, $C^\pi_{r_{10}}(w,x) \le C^\pi_{r'}(w,x)$ for all $r' \in \mathcal{R}^\pi$ and $w \in [0.12, \infty)$, and thus w^π_{10} is chosen as $w^\pi_{10} = \overline{w}^\pi = 0.14$. □

2. Related Literature

Dafermos (1972) considered a BiTAP with a finite number of user types. It was pointed out that it is a special case of the traffic assignment problem in which t_a depends on x and not just on x_a, that is, the travel cost on an arc a depends on the flow on other arcs and not just the flow on arc a. The results of Dafermos (1971) were used to obtain the following sufficient condition for the user-equilibrium to be given by a solution of an optimization problem: Let $w = 1, \dots, W$ index the user types, let $x^{(u,v)}_a(w)$ denote the flow of type w users from origin u to destination v on arc a, and let $\mathcal{X}$ denote the set of arc flows $(x^{(u,v)}_a(w), (u,v) \in \Pi, a \in \mathcal{A}, w = 1, \dots, W) \ge 0$ such that

$$\sum_{\{j \in \mathcal{N} : (i,j) \in \mathcal{A}\}} x^{(u,v)}_{(i,j)}(w) - \sum_{\{j \in \mathcal{N} : (j,i) \in \mathcal{A}\}} x^{(u,v)}_{(j,i)}(w) = \begin{cases} D^{(u,v)}(w) & \text{if } i = u \\ -D^{(u,v)}(w) & \text{if } i = v \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, i \in \mathcal{N},\ \forall\, (u,v) \in \Pi,\ \forall\, w = 1, \dots, W \tag{1}$$

Let $x_a(w) := \sum_{(u,v)\in\Pi} x_a^{(u,v)}(w)$, and let $x_a := (x_a(1),\ldots,x_a(W))$. Let

$$\overline{\mathcal{X}} := \left\{ (x_a, a\in\mathcal{A}) : x_a(w) = \sum_{(u,v)\in\Pi} x_a^{(u,v)}(w),\ \left(x_a^{(u,v)}(w), (u,v)\in\Pi, a\in\mathcal{A}, w=1,\ldots,W\right)\in\mathcal{X} \right\}$$

denote the set of feasible arc flows for each user type. Let $c_a(w,x_a)$ denote the unit travel cost of type w flow on arc a, and let $c(x) := (c_a(w,x_a), a\in\mathcal{A}, w=1,\ldots,W)$. Then user equilibrium can be specified with the variational inequality $x^*\in\overline{\mathcal{X}}$, $\langle c(x^*), y-x^*\rangle \geq 0$ for all $y\in\overline{\mathcal{X}}$, that is, $-c(x^*)\in\mathcal{N}_{\overline{\mathcal{X}}}(x^*)$, where $\mathcal{N}_{\overline{\mathcal{X}}}(x^*)$ denotes the normal cone to $\overline{\mathcal{X}}$ at x^*.

If, for each arc $a\in\mathcal{A}$ and all flow vectors x_a, the matrix with entries $\partial c_a(w,x_a)/\partial x_a(w')$, $w,w'=1,\ldots,W$, is symmetric and positive definite, then the user-equilibrium can be computed by solving

$$\min_{x\geq 0}\left\{ \sum_{a\in\mathcal{A}}\sum_{w=1}^{W}\int_0^{x_a(w)} c_a\big(w,(z,x_{-a}(-w))\big)\,\mathrm{d}z \ :\ (1)\right\}$$

where $x_{-a}(-w) := \big(x_{a'}(w'), a'\in\mathcal{A}, w'=1,\ldots,W, (a',w')\neq(a,w)\big)$ denotes the vector of all $x_{a'}(w')$ except for $(a',w')=(a,w)$. Note that in the case of the BiTAP, $c_a(w,x_a) = t_a(x_a(1)+\cdots+x_a(W)) + ws_a$, and thus $\partial c_a(w,x_a)/\partial x_a(w') = \mathrm{d}t_a(x_a(1)+\cdots+x_a(W))/\mathrm{d}x$. Hence, all entries $\partial c_a(w,x_a)/\partial x_a(w')$ of the matrix are the same positive number $\mathrm{d}t_a(x_a(1)+\cdots+x_a(W))/\mathrm{d}x$, and therefore the matrix for the BiTAP is symmetric and positive semi-definite. Dafermos (1972) proposed an algorithm that solves shortest path problems similar to the Frank-Wolfe algorithm for the traffic assignment problem, but instead of the new point being a convex combination of the previous point and the shortest path, the new point is determined by removing flow from the most expensive path with positive flow for an OD pair, and adding the flow to the shortest path for the OD pair.

Dial (1979) considered the special case of the BiTAP in which both s_a and t_a do not depend on the flow rates x. An algorithm was proposed that, for a given OD pair π, finds a set $\{r_1,\ldots,r_k\}\subset\mathcal{R}^\pi$ of efficient paths, and corresponding thresholds $\underline{w}^\pi = w_0^\pi < w_1^\pi < \cdots < w_{k-1}^\pi < w_k^\pi = \overline{w}^\pi$, such that for all $i=1,\ldots,k$ it holds that $C_{r_i}^\pi(w,x)\leq C_{r'}^\pi(w,x)$ for all $r'\in\mathcal{R}^\pi$ and $w\in[w_{i-1}^\pi, w_i^\pi]$.

Daganzo (1983) considered a traffic assignment problem with a finite number of user types, similar to Dafermos (1972). The model allows more general arc costs of the form $c_a(w,x) := t_a(x) + ws_a$ that may depend on the flows x on all arcs. It is assumed that the mapping $t:\mathbb{R}^{\mathcal{A}}\mapsto\mathbb{R}^{\mathcal{A}}$ is increasing. In addition, it is assumed that for each user type $w=1,\ldots,W$, there is a function $x^w:\mathbb{R}^{\mathcal{A}}\mapsto\mathbb{R}^{\mathcal{A}}$ that maps arc costs to arc flows that is continuously differentiable with a negative semidefinite Jacobian. The assumption that x^w is continuous means that the Wardrop equilibrium

conditions (only OD paths with least travel cost have positive flow) do not apply. Existence of a fixed point is established with Brouwer's fixed point theorem and Kakutani's fixed point theorem. A stochastic approximation algorithm is proposed to compute fixed points.

Leurent (1993) considered the following BiTAP. (Leurent (1993) groups paths $r \in \mathcal{R}^\pi$ with the same value of S_r^π into "monetary expense classes", but to simplify notation we omit the classification.) For each OD pair $\pi \in \Pi$, and each path $r \in \mathcal{R}^\pi$, let $F_r^\pi := \sum_{r'=1}^{r} f_{r'}^\pi$ denote the cumulative flow on paths $r' = 1, \ldots, r$, let $F_0^\pi := 0$, and let $F^\pi := \sum_{r \in \mathcal{R}^\pi} f_r^\pi = F_{R^\pi}^\pi$ denote the total flow for OD pair π. It is assumed that G^π has a positive density so that $(G^\pi)^{-1}(q)$ is a singleton for all $q \in (0,1)$. The OD demand is allowed to depend on the mean total cost over all OD paths and all weights, as follows:

$$F^\pi \;=\; D^\pi\left(\sum_{r\in\mathcal{R}^\pi}\left[\frac{f_r^\pi}{F^\pi}T_r^\pi(x) + S_r^\pi\int_{F_{r-1}^\pi/F^\pi}^{F_r^\pi/F^\pi}(G^\pi)^{-1}(q)\,\mathrm{d}q\right]\right)$$

for $F^\pi > 0$, where $D^\pi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is assumed to be continuous and decreasing. It is shown that the equilibrium conditions coincide with the first-order optimality conditions of the following optimization problem:

$$\min_{f\geq 0}\left\{\sum_{a\in\mathcal{A}}\int_0^{x_a} t_a(z)\,\mathrm{d}z + \sum_{\pi\in\Pi}F^\pi\sum_{r\in\mathcal{R}^\pi}S_r^\pi\int_{F_{r-1}^\pi/F^\pi}^{F_r^\pi/F^\pi}(G^\pi)^{-1}(q)\,\mathrm{d}q - \sum_{\pi\in\Pi}\int_0^{F^\pi}(D^\pi)^{-1}(q)\,\mathrm{d}q\right\} \quad (2)$$

It is shown that the objective function is convex. Similar to Daganzo (1983), a stochastic approximation algorithm is proposed to solve the optimization problem. The algorithm enumerates paths.

Leurent (1996) considered the same problem as in Leurent (1993), with the possible addition of arc capacity constraints of the form $x_a \leq C_a$ (here, C_a denotes the capacity of arc a). Three algorithms are compared for the case without capacity constraints. For the case with capacity constraints, an augmented Lagrangian algorithm is proposed.

Marcotte and Zhu (1996) considered the following formulation of the BiTAP. It is assumed that G^π has a density, but as before we give a more general formulation here to match our notation. Let $x_a^\pi(w)$ denote the (unscaled) flow on arc $a \in \mathcal{A}$ of travelers for OD pair $\pi \in \Pi$ with preference weight $w \in [0, w_{\max}]$. Also, for each arc $a \in \mathcal{A}$, let $X_a := \sum_{\pi\in\Pi}\int_0^{w_{\max}} x_a^\pi(w)\,\mathrm{d}G^\pi(w)$ denote the total flow on arc a. Let $\mathcal{L}_2^\pi$ denote the set of functions $f : [0, w_{\max}] \mapsto \mathbb{R}$ such that $\int_0^{w_{\max}} f^2(w)\,\mathrm{d}G^\pi(w) < \infty$; for any $f^\pi, g^\pi \in \mathcal{L}_2^\pi$, let $\langle f^\pi, g^\pi\rangle_2^\pi := \int_0^{w_{\max}} f^\pi(w)g^\pi(w)\,\mathrm{d}G^\pi(w)$. Then, for any $f = \left(f_a^\pi \in \mathcal{L}_2^\pi, \pi\in\Pi, a\in\mathcal{A}\right) \in \mathcal{L}_2 := \prod_{\pi\in\Pi}\left(\mathcal{L}_2^\pi\right)^{\mathcal{A}}$, $g = \left(g_a^\pi \in \mathcal{L}_2^\pi, \pi\in\Pi, a\in\mathcal{A}\right) \in \mathcal{L}_2$, let $\langle f, g\rangle_2 := \sum_{\pi\in\Pi}\sum_{a\in\mathcal{A}}\langle f_a^\pi, g_a^\pi\rangle_2^\pi$. For any $x \in \mathcal{X}$, let $c_a^\pi(w,x) := t_a(X_a) + ws_a$ (it has the same value

for all $\pi \in \Pi$), and let $c(x) := (c_a^\pi(\cdot, x) \in \mathcal{L}_2^\pi, \pi \in \Pi, a \in \mathcal{A}) \in \mathcal{L}_2$. Then the equilibrium conditions can be expressed as the infinite dimensional variational inequality $x^* \in \mathcal{X}$, $\langle c(x^*), y - x^* \rangle_2 \geq 0$ for all $y \in \mathcal{X}$. It is shown that the equilibrium conditions coincide with the first-order optimality conditions of the infinite dimensional optimization problem

$$\min_{x \in \mathcal{X}} \left\{ \sum_{a \in \mathcal{A}} \int_0^{X_a} t_a(z)\, \mathrm{d}z + \sum_{a \in \mathcal{A}} s_a \sum_{\pi \in \Pi} \int_{\underline{w}}^{\overline{w}} w x_a^\pi(w)\, \mathrm{d}G^\pi(w) \right\} \tag{3}$$

It is shown that the objective function is convex. The Frank-Wolfe subproblem at $\bar{x} \in \mathcal{X}$ is

$$\min_{x \in \mathcal{X}} \sum_{a \in \mathcal{A}} \sum_{\pi \in \Pi} \int_{\underline{w}}^{\overline{w}} \left[t_a(\bar{X}_a) + w s_a \right] x_a^\pi(w)\, \mathrm{d}G^\pi(w) \tag{4}$$

The Frank-Wolfe subproblem separates into the following shortest path problem for each $(u, v) \in \Pi$ and each $w \in [\underline{w}, \overline{w}]$:

$$\min_{x^{(u,v)}(w) \geq 0} \left\{ \sum_{a \in \mathcal{A}} \left[t_a(\bar{X}_a) + w s_a \right] x_a^{(u,v)}(w) \ : \ (1) \right\} \tag{5}$$

It is mentioned that, for each $(u, v) \in \Pi$, problems (5) for all $w \in [0, w_{\max}]$ can be solved by solving one parametric shortest path problem with parameter w. Thereafter a line search problem is solved to determine the minimizing convex combination between the previous flows x and the shortest path flows obtained from the Frank-Wolfe subproblem. Marcotte et al. (1996) provided more details about the proposed algorithm, as well as numerical results.

Dial (1996) considered the BiTAP in which both arc costs s_a and t_a are allowed to be functions of the total arc flow x_a. Two algorithms were proposed. One algorithm is a heuristic based on the efficient paths algorithm of Dial (1979) and Frank-Wolfe steps. Another algorithm is based on the same optimization problem as in Marcotte and Zhu (1996) and Marcotte et al. (1996), with s_a not varying with the arc flow x_a. Dial (1997) considered the same problem as in Dial (1996). An algorithm is proposed to solve the same variational inequality, $x^* \in \mathcal{X}$, $\langle c(x^*), y - x^* \rangle_2 \geq 0$ for all $y \in \mathcal{X}$, as in Marcotte and Zhu (1996). Instead of solving the optimization problem (3), the proposed algorithm alternates between solving the parametric shortest path problem (5) at current flow $\bar{x} \in \mathcal{X}$ that produces flow $x^m \in \mathcal{X}$ in iteration m, and a linear programming approximation that chooses the next flow as the convex combination $x(\lambda) := \sum_{m'=1}^m \lambda_{m'} x^{m'}$, $\sum_{m'=1}^m \lambda_{m'} = 1$, $\lambda \geq 0$, attempting to minimize $\sum_{m'=1}^m \max\left\{0, \left\langle c\left(x(\lambda)\right), x^{m'} - x(\lambda) \right\rangle_2\right\}$.

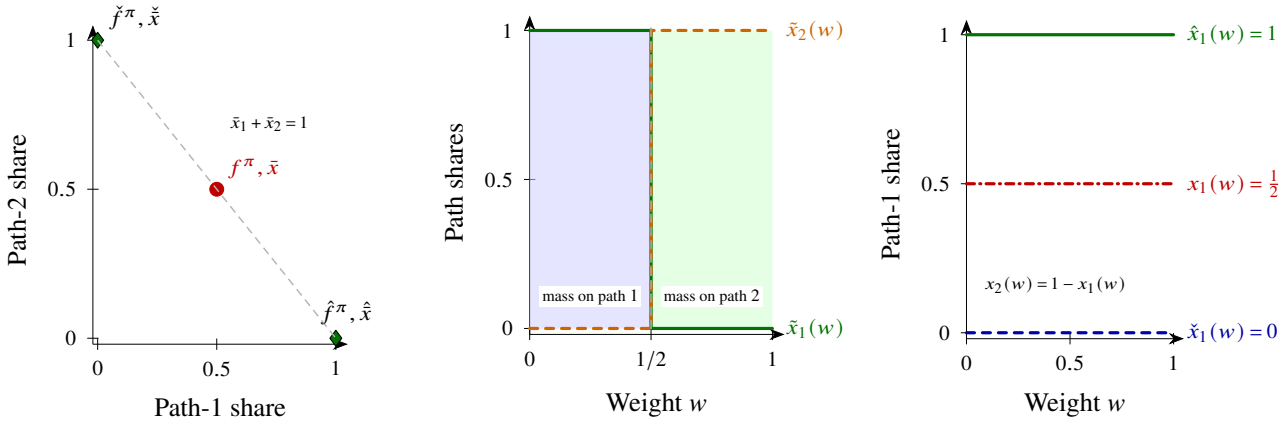
Marcotte and Zhu (1997) considered the problem in Marcotte and Zhu (1996) and Marcotte et al. (1996) with more general linear constraints and with arc costs given by $c_a^\pi(x)(w) = t_a(X) +$

$ws_a(X)$. Marcotte and Zhu (1997) showed that, if s and t are Lipschitz continuous and continuously differentiable, then c is a monotone mapping if and only if $t : \mathbb{R}^{\mathcal{A}} \mapsto \mathbb{R}^{\mathcal{A}}$ is monotone and $s : \mathbb{R}^{\mathcal{A}} \mapsto \mathbb{R}^{\mathcal{A}}$ is constant.

Marcotte (1999) compared a number of formulations for the BiTAP. It was assumed that G^π has a positive density g on $[\underline{w}, \overline{w}]$ for all π. Marcotte (1999) showed that the formulations in Leurent (1993) (with fixed demand D^π and with the ratios f_r^π / F^π replaced by $\bar{x}_r$) and in Marcotte and Zhu (1996) and Marcotte et al. (1996) are equivalent. However, it is important to note that the Frank-Wolfe framework applied to the formulations does not give equivalent algorithms. The reason is that convex combinations of equivalent solutions for different formulations do not give equivalent solutions. To illustrate that, consider the following simple example.

EXAMPLE 2. The network has a single OD pair π with fixed demand $D^\pi = 1$, two paths indexed 1 and 2 from the origin to the destination with $1 \prec 2$, and weights w are uniformly distributed on $[\underline{w}^\pi, \overline{w}^\pi] = [0, 1]$ for both paths. Solution 1 puts all flow on path 1. Solution 1 for the formulations of Leurent (1993) and Marcotte (1999) is depicted by the bottom right diamond in Figure 2(a), and solution 1 for the formulation of Marcotte and Zhu (1996) and Marcotte et al. (1996) is depicted by the top horizontal line in Figure 2(c). Solution 2 puts all flow on path 2. Solution 2 for the formulations of Leurent (1993) and Marcotte (1999) is depicted by the top left diamond in Figure 2(a), and solution 2 for the formulation of Marcotte and Zhu (1996) and Marcotte et al. (1996) is depicted by the bottom horizontal line in Figure 2(c).

Next, consider the convex combination halfway between the two solutions. The convex combination for the formulations of Leurent (1993) and Marcotte (1999) is depicted by the middle dot in Figure 2(a), and the convex combination for the formulation of Marcotte and Zhu (1996) and Marcotte et al. (1996) is depicted by the middle horizontal line in Figure 2(c). Also, recall from Example 1 and the preceding explanation that, given solution f^π or $\bar{x}$ and taking into account that $1 \prec 2$, there are thresholds $\underline{w}^\pi = w_0^\pi \le w_1^\pi \le w_2^\pi = \overline{w}^\pi$, such that users with weight $w \in (w_{i-1}^\pi, w_i^\pi)$ travel on path i and thus contribute to flow f_i^π or $\bar{x}_i$. Thus, solution f^π or $\bar{x}$ can be extended to weight-dependent flow functions $\tilde{x}_i(w)$. (This statement is based on the distribution G^π having a density, and we will show how this construction can be generalized for arbitrary distributions.) Figure 2(b) shows the resulting weight-dependent flow functions $\tilde{x}_1(w) = 1$ and $\tilde{x}_2(w) = 0$ for all $w \in (0, 1/2)$ and $\tilde{x}_1(w) = 0$ and $\tilde{x}_2(w) = 1$ for all $w \in (1/2, 1)$. Note that the weight-dependent flow functions $\tilde{x}$ resulting from the convex combination $(1-\alpha)\hat{f}^\pi + \alpha\check{f}^\pi$ is not equivalent to the convex combination $(1-\alpha)\hat{x} + \alpha\check{x}$ depicted by the middle horizontal line in Figure 2(c). Also note that



(a) Solutions and convex combination for formulations of Leurent (1993) and Marcotte (1999).

(b) Weight-dependent flow solution implied by convex combination for formulations of Leurent (1993) and Marcotte (1999).

(c) Solutions and convex combination for formulation of Marcotte and Zhu (1996) and Marcotte et al. (1996).

Figure 2 Difference between convex combinations for aggregate-flow formulations and convex combinations for function-valued formulations for Example 2.

the weight-dependent flow functions $\tilde{x}$ has the interval structure illustrated in Example 1 and the preceding explanation, while no strict convex combination $(1-\alpha)\hat{x} + \alpha\check{x}$ has the interval structure. This difference has important computational consequences. □

Marcotte (1999) also considered the following formulation for the BiTAP based on thresholds $\underline{w}^\pi = w_0^\pi \le w_1^\pi \le \cdots \le w_{R^\pi-1}^\pi \le w_{R^\pi}^\pi = \overline{w}^\pi$. Given $w = (w_r^\pi, \pi \in \Pi, r \in \mathcal{R}^\pi)$, the flow on each path $r \in \mathcal{R}^\pi$ is equal to $f_r^\pi(w^\pi) = D^\pi \int_{w_{r-1}^\pi}^{w_r^\pi} g^\pi(z)\,\mathrm{d}z$, and the flow on each arc $a \in \mathcal{A}$ is given by $x_a = \sum_{\pi\in\Pi} \sum_{r\in\mathcal{R}_a^\pi} f_r^\pi(w^\pi)$. Assuming that costs S_r^π do not depend on the flows, and with thresholds as decision variables, the equilibrium flows are given by the optimal solutions of the optimization problem

$$\min_{w} \left\{ \sum_{a\in\mathcal{A}} \int_0^{x_a} t_a(z)\,\mathrm{d}z + \sum_{\pi\in\Pi} D^\pi \sum_{r\in\mathcal{R}^\pi} S_r^\pi \int_{w_{r-1}^\pi}^{w_r^\pi} z g^\pi(z)\,\mathrm{d}z \right\} \tag{6}$$

$$\text{s.t. } w_r^\pi - w_{r-1}^\pi \ge 0 \qquad \forall\, \pi \in \Pi,\ r \in \mathcal{R}^\pi \tag{7}$$

$$w_0^\pi = \underline{w}, \qquad w_{R^\pi}^\pi = \overline{w} \qquad \forall\, \pi \in \Pi \tag{8}$$

As pointed out by Marcotte (1999), one issue with this formulation is that the objective is not convex in w (the variational inequality is not monotone in w). On the positive side, it is shown that a point that satisfies the first-order optimality conditions of (6)–(8) is an optimal solution of (6)–(8).

Xie et al. (2026) considered the formulation (6)–(8), and confirmed that the first-order optimality conditions for (6)–(8) coincide with the equilibrium conditions for the BiTAP. Similar to Marcotte

and Zhu (1996) and Marcotte et al. (1996), Xie et al. (2026) used a parametric shortest path algorithm to generate efficient paths. Unlike previously proposed algorithms based on Frank-Wolfe, instead of performing a line search between the current solution and an optimal solution of the Frank-Wolfe subproblem, the Single-Boundary Adjustment (SBA) algorithm proposed by Xie et al. (2026) adjusted one threshold w_r^π at a time using an Armijo line search algorithm.

3. Alternative Formulation of the Bicriteria Traffic Assignment Problem

Recall that the formulation (3) considered in Marcotte and Zhu (1996) and Marcotte et al. (1996) is a convex optimization problem, but that it has infinite dimensional decision variables (weight-dependent flow functions), and that even if two solutions have the interval structure possessed by optimal solutions, a strict convex combination of the two solutions loses the interval structure. These properties are detrimental to efficient computation. In contrast, the formulation (6)–(8) considered in Marcotte (1999) and Xie et al. (2026) has finite dimensional decision variables that maintain the interval structure by construction, but the objective function is not convex. In this section, we propose an alternative formulation. Similar to the formulation (2) considered in Leurent (1993) and Leurent (1996), our formulation is a convex optimization problem with finite dimensional decision variables. All the previous formulations either require G^π to have finite support or require G^π to have a positive density on $[\underline{w}^\pi, \overline{w}^\pi]$ for each $\pi \in \Pi$. In contrast, our formulation does not require anything of G^π. To simplify the presentation and algorithm, we also assume that G^π has bounded support; this assumption can be removed at the cost of complicating the algorithm.

3.1. Notation

For a convex set $S \subset \mathbb{R}^n$, the relative interior of S is denoted by $\mathrm{ri}(S)$, and the normal cone to S at x is denoted by $\mathcal{N}_S(x)$. Consider a convex function $\psi : \mathbb{R}^n \mapsto \mathbb{R} \cup \{\infty\}$. The effective domain of ψ is denoted by $\mathrm{dom}(\psi) := \{x \in \mathbb{R}^n : \psi(x) < \infty\}$. Let

$$\partial\psi(z) \;:=\; \{u \in \mathbb{R}^n \,:\, \psi(z') \geq \psi(z) + \langle u, z' - z\rangle \;\forall\, z' \in \mathbb{R}^n\} \tag{9}$$

denote the subdifferential of ψ at a point $z \in \mathrm{dom}(\psi)$. For a finite set $\mathcal{I}$, let $\mathbb{R}^{\mathcal{I}}$ denote the vector space indexed by $\mathcal{I}$. Let

$$\Delta_{\mathcal{I}} \;:=\; \left\{x \in \mathbb{R}_+^{\mathcal{I}} \,:\, \sum_{i\in\mathcal{I}} x_i = 1\right\}$$

denote the standard simplex over $\mathcal{I}$. Let

$$\Delta_{\mathcal{I}}^\circ \;:=\; \{x \in \Delta_{\mathcal{I}} \,:\, x_i > 0 \;\forall\, i \in \mathcal{I}\}$$

denote the relative interior of $\Delta_{\mathcal{I}}$. For any Borel set $S \subset \mathbb{R}$, let $\mathcal{P}(S)$ denote the set of cumulative distribution functions on S. The quantile map for any $G \in \mathcal{P}([\underline{w}, \overline{w}])$ is defined as the set-valued map $G^{-1} : [0,1] \mapsto 2^{\mathbb{R}}$ given by

$$G^{-1}(p) \;:=\; \{x \in \mathbb{R} \,:\, G(y) \le p \;\forall\, y < x \text{ and } G(y) \ge p \;\forall\, y > x\}.$$

Note that $G^{-1}(p)$ is a nonempty closed interval for all $p \in (0,1)$, and $G^{-1}(0) = (-\infty, w]$ and $G^{-1}(1) = [w', \infty)$ for $\underline{w} \le w \le w' \le \overline{w}$. The quantile map is non-decreasing in the sense that for any $0 \le a < b \le 1$, $x_a \in G^{-1}(a)$, and $x_b \in G^{-1}(b)$, it holds that $x_a \le x_b$.

For any $0 \le a \le b \le 1$, let the truncation of $G \in \mathcal{P}([\underline{w}, \overline{w}])$ be given by

$$G_{a,b}(x) \;:=\; \max\{a, \min\{b, G(x)\}\}.$$

Note that $G_{a,b}$ is nondecreasing and right-continuous, and it induces a finite Lebesgue-Stieltjes measure on $\mathbb{R}$, which we also denote by $G_{a,b}$. Its total mass satisfies

$$\int_{\mathbb{R}} \mathrm{d}G_{a,b} \;=\; \lim_{x\to\infty} G_{a,b}(x) - \lim_{x\to-\infty} G_{a,b}(x) \;=\; b - a.$$

Note that for any integrable Borel measurable function $f : \mathbb{R} \mapsto \mathbb{R}$ and any $0 \le a \le b \le c \le 1$, the additivity property

$$\int_{\mathbb{R}} f(x)\; \mathrm{d}G_{a,c}(x) \;=\; \int_{\mathbb{R}} f(x)\; \mathrm{d}G_{a,b}(x) + \int_{\mathbb{R}} f(x)\; \mathrm{d}G_{b,c}(x) \tag{10}$$

holds.

3.2. Formulation

The variables q_r^π denote the fraction of demand D^π for OD pair $\pi \in \Pi$ that flows on path $r \in \mathcal{R}^\pi$. Let $q := \left(q_r^\pi, \pi \in \Pi, r \in \mathcal{R}^\pi\right)$. As before, the weight thresholds are denoted with $w = (w_r^\pi, \pi \in \Pi, r \in \mathcal{R}^\pi)$. Recall that for each OD pair $\pi \in \Pi$, we choose a total order $\prec$ on the set of paths $\mathcal{R}^\pi$ such that for any $r, r' \in \mathcal{R}^\pi$ it holds that if $r \prec r'$ (implying $r \ne r'$) then $S_r^\pi \ge S_{r'}^\pi$. To make expressions more concise, for every $\pi \in \Pi$, and $r \in \mathcal{R}^\pi$, let

$$p_r^\pi \;:=\; \sum_{\{r' \in \mathcal{R}^\pi \,:\, r' \preceq r\}} q_{r'}^\pi$$

denote the cumulative fraction of demand D^π for OD pair $\pi \in \Pi$ that flows on paths $r' \in \mathcal{R}^\pi$ with $r' \preceq r$. Also, let the linear operator $L^\pi : \mathbb{R}^{\mathcal{R}^\pi} \mapsto \mathbb{R}^{\mathcal{R}^\pi}$ be given by

$$L_r^\pi(q) \;:=\; \sum_{\{r' \in \mathcal{R}^\pi \,:\, r' \preceq r\}} q_{r'} \;=\; (\mathbb{1}\{r' \preceq r\},\, r' \in \mathcal{R}^\pi)^\top\, (q_{r'},\, r' \in \mathcal{R}^\pi)$$

so that $p^\pi = L^\pi(q^\pi)$. Also, for each $r \in \mathcal{R}^\pi$, $r-1$ denotes the path that is the immediate predecessor of r, so that $p^\pi_{r-1} := \sum_{\{r' \in \mathcal{R}^\pi : r' \prec r\}} q^\pi_{r'}$, with the convention that if r has no predecessor in $\mathcal{R}^\pi$, then $p^\pi_{r-1} = 0$ and $w^\pi_{r-1} = \underline{w}^\pi$. Given q, for each arc $a \in \mathcal{A}$, let

$$x_a(q) \;:=\; \sum_{\pi \in \Pi} D^\pi \sum_{r \in \mathcal{R}^\pi_a} q^\pi_r \qquad \forall\, a \in \mathcal{A} \tag{11}$$

denote the total flow on arc a, and let $x(q) := (x_a(q), a \in \mathcal{A})$.

3.2.1. Equilibrium Conditions The pair (q, w) satisfies the equilibrium conditions if the following hold for all $\pi \in \Pi$:

1.
$$q^\pi \;\in\; \Delta_{\mathcal{R}^\pi}, \quad \text{i.e.} \sum_{r \in \mathcal{R}^\pi} q^\pi_r \;=\; 1 \;\text{ and }\; q^\pi_r \;\geq\; 0 \;\forall\, r \in \mathcal{R}^\pi \tag{12}$$
that is, each q^π is a probability distribution with support on $\mathcal{R}^\pi$.

2.
$$\underline{w}^\pi \leq \cdots \leq w^\pi_{r-1} \leq w^\pi_r \leq \cdots \leq \overline{w}^\pi \tag{13}$$
that is, the thresholds w^π_r are properly ordered between $\underline{w}^\pi$ and $\overline{w}^\pi$. We can also require that the last threshold be equal to $\overline{w}^\pi$, but given the other conditions, it is not necessary.

3. For each $r \in \mathcal{R}^\pi$ such that $q^\pi_r > 0$, it holds that
$$C^\pi_r(w, x(q)) \;\leq\; C^\pi_{r'}(w, x(q)) \quad \forall\, r' \in \mathcal{R}^\pi, \quad \forall\, w \in [w^\pi_{r-1}, w^\pi_r] \tag{14}$$
that is, each used path is user-optimal for all weights w in its interval.

4.
$$w^\pi_r \;\in\; (G^\pi)^{-1}\left(L^\pi_r(q^\pi)\right) \quad \forall\, r \in \mathcal{R}^\pi \tag{15}$$
that is, the thresholds and probabilities are consistent with each other.

EXAMPLE 3. Consider Example 1, and suppose that OD pair π with demand $D^\pi = 1$ is the only OD pair with positive demand. Let

$$G^\pi(w) \;=\; \begin{cases} 0 & \text{if } w < 0 \\ 0.05 & \text{if } 0 \leq w < 0.01 \\ 0.3 + 5\,(w - 0.01) & \text{if } 0.01 \leq w < 0.07 \\ 0.8 & \text{if } 0.07 \leq w < 0.14 \\ 1 & \text{if } w \geq 0.14 \end{cases}$$

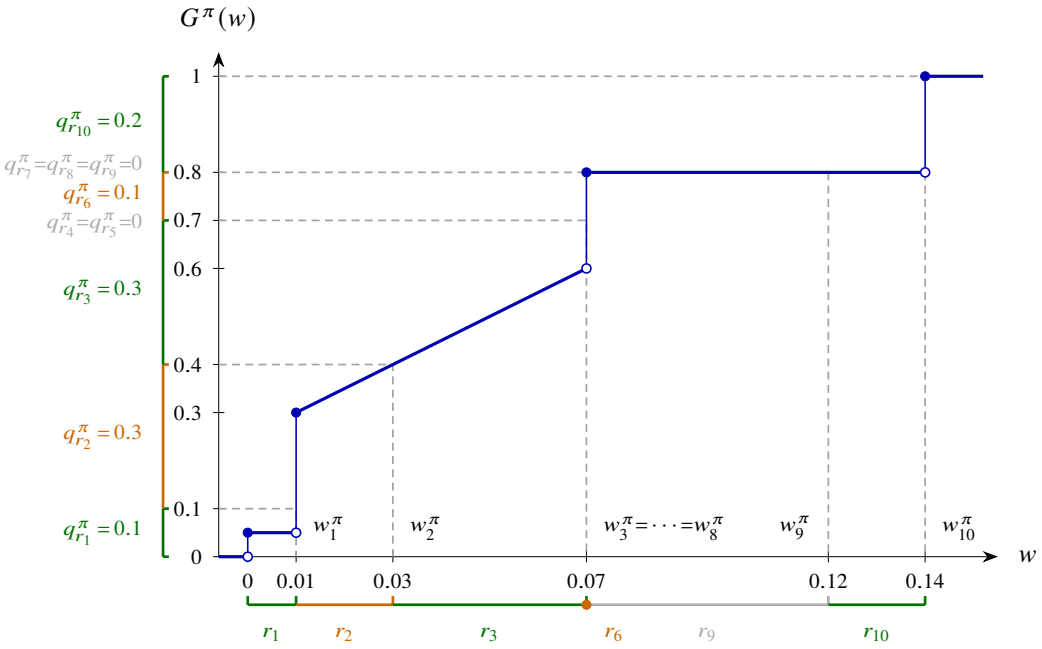


Figure 3 The distribution G^π of Example 3, the fractions q^π_r as brackets to the left of the vertical axis, the cumulative fractions p^π_r on the vertical axis, the thresholds w^π_r on the horizontal axis, and the intervals $[w^\pi_{r-1}, w^\pi_r]$ as brackets below the horizontal axis. Dashed lines connect each cumulative fraction p^π_r to the thresholds $w^\pi_r \in (G^\pi)^{-1}(p^\pi_r)$ of condition (15).

as shown in Figure 3. Let $\left(q^\pi_{r_1}, \ldots, q^\pi_{r_{10}}\right) = (0.1, 0.3, 0.3, 0, 0, 0.1, 0, 0, 0, 0.2)$, so that condition (12) holds and $\left(p^\pi_{r_1}, \ldots, p^\pi_{r_{10}}\right) = L^\pi(q^\pi) = (0.1, 0.4, 0.7, 0.7, 0.7, 0.8, 0.8, 0.8, 0.8, 1)$. Figure 3 shows how q^π partitions the demand among the used paths. Consider $\left(w^\pi_1, \ldots, w^\pi_{10}\right) = (0.01, 0.03, 0.07, 0.07, 0.07, 0.07, 0.07, 0.07, 0.12, 0.14)$, which are the thresholds of Example 1 with the choice $\eta = 0.03$, so that conditions (13) and (14) hold. Figure 3 shows that condition (15) also holds, and thus (q, w) is an equilibrium. Recall that paths r_2 and r_3 are tied. It is clear how their order could be swapped and thus (q, w) is not a unique equilibrium. Figure 3 exhibits some features excluded by formulations that require a positive density and distinct costs: paths r_1 and r_2 split the jump at w^π_1, paths r_3 and r_6 split the jump at $w^\pi_3 = \cdots = w^\pi_8$, path r_6 carries positive flow although its interval is a singleton, and path r_9 carries zero flow although its interval $[0.07, 0.12]$ is not a singleton. The example does not show how $T^\pi_r(x)$ specified in Example 1 is obtained from the travel time functions t_a and the flow partition q — that will be demonstrated in Example 6. □

3.2.2. Optimization Problem In this section we formulate a convex optimization problem that we use to compute an equilibrium. In a typical solution of the traffic assignment problem, only a small fraction of the paths $r \in \mathcal{R}^\pi$ for an OD pair $\pi \in \Pi$ is used (has $q^\pi_r > 0$). For algorithmic

efficiency, it is important to maintain sparse solutions, and our proposed algorithm is designed to consider small subsets of paths $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$, $\pi \in \Pi$, and find the best solution for the subsets in each iteration. Therefore, we extend the notation to make provision for the smaller as well as the full optimization problems. For each $a \in \mathcal{A}$, let $\hat{\mathcal{R}}^\pi_a := \hat{\mathcal{R}}^\pi \cap \mathcal{R}^\pi_a$ denote the paths in $\hat{\mathcal{R}}^\pi$ that use arc a. Note that, for each $\pi \in \Pi$, the same order $\prec$ applies to every subset $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$. When a nonempty subset $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$ of paths is considered, we will let $q := \left(q^\pi_r, \pi \in \Pi, r \in \hat{\mathcal{R}}^\pi\right)$, we will require that $\sum_{r\in\hat{\mathcal{R}}^\pi} q^\pi_r = 1$ and

$$p^\pi_r \;:=\; \sum_{\{r'\in\hat{\mathcal{R}}^\pi\,:\,r'\preceq r\}} q^\pi_{r'},$$

and thus the linear operator $L^\pi : \mathbb{R}^{\hat{\mathcal{R}}^\pi} \mapsto \mathbb{R}^{\hat{\mathcal{R}}^\pi}$ will be given by

$$L^\pi_r(q) \;:=\; \sum_{\{r'\in\hat{\mathcal{R}}^\pi\,:\,r'\preceq r\}} q_{r'} \;=\; \left(\mathbb{1}\{r'\preceq r\},\, r'\in\hat{\mathcal{R}}^\pi\right)^\top \left(q_{r'},\, r'\in\hat{\mathcal{R}}^\pi\right)$$

so that $p^\pi = L^\pi(q^\pi)$. Also, for each $r \in \hat{\mathcal{R}}^\pi$, $r-1$ will denote the path that is the immediate predecessor in $\hat{\mathcal{R}}^\pi$ of r, so that $p^\pi_{r-1} := \sum_{\{r'\in\hat{\mathcal{R}}^\pi\,:\,r'\prec r\}} q^\pi_{r'}$. (The subset $\hat{\mathcal{R}}^\pi$ will be clear from the context, and therefore is not displayed in the notation.) Then, let $\phi : \prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi} \mapsto \mathbb{R}$ be given by

$$\phi(q) \;:=\; \sum_{\pi\in\Pi} D^\pi \sum_{r\in\hat{\mathcal{R}}^\pi} S^\pi_r \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{p^\pi_{r-1},p^\pi_r}(y) \tag{16}$$

where, in this formulation, variables p^π_r are not regarded as separate decision variables, but rather as linear functions of q. Also, let $\psi : \prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi} \mapsto \mathbb{R}$ be given by

$$\psi(q) \;:=\; \sum_{a\in\mathcal{A}} \int_0^{x_a(q)} t_a(y)\; \mathrm{d}y \tag{17}$$

where

$$x_a(q) \;:=\; \sum_{\pi\in\Pi} D^\pi \sum_{r\in\hat{\mathcal{R}}^\pi_a} q^\pi_r \qquad \forall\, a\in\mathcal{A}$$

denotes the total flow on arc a. For any subsets $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$, $\pi \in \Pi$, the optimization problem is as follows:

$$\min_q \left\{ \Upsilon(q) \;:=\; \psi(q) + \phi(q) \;:\; q \in \prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi} \right\} \tag{18}$$

A similar formulation can be written with p^π_r as decision variables, with no apparent advantage of one formulation over the other. Note that, whereas the equilibrium conditions are stated for a pair (q, w), a solution of the optimization problem is q only. Thus, unlike previous approaches, the

first-order optimality conditions of the optimization problem do not coincide with the equilibrium conditions. Instead, we will show how, given a solution q of the optimization problem for $\hat{\mathcal{R}}^\pi = \mathcal{R}^\pi$, $\pi \in \Pi$, thresholds w can be determined in run time that is linear in the number of efficient paths, such that (q, w) is an equilibrium.

4. The Optimization Problem and Equilibria

4.1. Preliminary Results

First we confirm that the value of $\phi(q)$ is not affected by the choice of total order on $\mathcal{R}^\pi$.

LEMMA 1. *For any $\pi \in \Pi$, consider two total orders $\prec$ and $\lessdot$ on $\mathcal{R}^\pi$ such that for any $r, r' \in \mathcal{R}^\pi$ it holds that if $r \prec r'$ or $r \lessdot r'$ then $S^\pi_r \geq S^\pi_{r'}$. Consider any $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$ and any $q^\pi \in \Delta_{\hat{\mathcal{R}}^\pi}$. For each $r \in \hat{\mathcal{R}}^\pi$, let*

$$p^\pi_r \;:=\; \sum_{\{r' \in \hat{\mathcal{R}}^\pi \,:\, r' \preceq r\}} q^\pi_{r'} \qquad \text{and} \qquad \dot{p}^\pi_r \;:=\; \sum_{\{r' \in \hat{\mathcal{R}}^\pi \,:\, r' \lessdot r\}} q^\pi_{r'}$$

Then

$$\sum_{r \in \hat{\mathcal{R}}^\pi} S^\pi_r \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{p^\pi_{r-1}, p^\pi_r}(y) \;=\; \sum_{r \in \hat{\mathcal{R}}^\pi} S^\pi_r \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{\dot{p}^\pi_{r-1}, \dot{p}^\pi_r}(y)$$

and thus $\phi(q)$ is invariant with respect to the chosen order.

Next we establish some pertinent properties of the function ϕ. For that purpose, consider any $[\underline{w}, \overline{w}] \subset \mathbb{R}_+$, any $G \in \mathcal{P}([\underline{w}, \overline{w}])$, and the function $\varphi : [0,1] \to \mathbb{R}$ given by

$$\varphi(p) \;:=\; \int_{\mathbb{R}} w \; \mathrm{d}G_{0,p}(w)$$

It follows from $[\underline{w}, \overline{w}] \subset \mathbb{R}_+$ that φ is nondecreasing. Note that $\varphi(0) = 0$ and $\varphi(1) \in [\underline{w}, \overline{w}]$. We will also use the result in Lemma 2.

LEMMA 2. *The function $\varphi : [0,1] \to \mathbb{R}$ is convex and Lipschitz continuous, and $\partial\varphi(p) = G^{-1}(p)$ for every $p \in [0,1]$.*

For every $\pi \in \Pi$, let $\varphi^\pi : [0,1] \to \mathbb{R}$ be given by

$$\varphi^\pi(p) \;:=\; \int_{\mathbb{R}} w \; \mathrm{d}G^\pi_{0,p}(w)$$

and given any subset of paths $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$ and any $r \in \hat{\mathcal{R}}^\pi$, let $\phi^\pi_r : \Delta_{\hat{\mathcal{R}}^\pi} \to \mathbb{R}$ be given by

$$\phi^\pi_r(q) \;:=\; \varphi^\pi\big(L^\pi_r(q)\big)$$

Then it follows from Lemma 2 that $\phi_r^\pi : \Delta_{\hat{\mathcal{R}}^\pi} \to \mathbb{R}$ is convex and Lipschitz continuous. Also, $\text{range}(L_r^\pi) \cap \text{ri}\big(\text{dom}(\varphi^\pi)\big) = (0,1) \neq \varnothing$, and thus it follows from Rockafellar (1997, Theorem 23.9) that

$$\partial \phi_r^\pi(q) \;=\; \left(\mathbb{1}\left\{r' \preceq r\right\},\, r' \in \hat{\mathcal{R}}^\pi\right) (G^\pi)^{-1}\left(L_r^\pi(q)\right) + \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q)$$

Let $S_{\min}^\pi := \min\{S_r^\pi \,:\, r \in \hat{\mathcal{R}}^\pi\}$. For each $r \in \hat{\mathcal{R}}^\pi$ that has a successor in $\hat{\mathcal{R}}^\pi$ in the order $\prec$, let $r+1 \in \hat{\mathcal{R}}^\pi$ denote the immediate successor in $\hat{\mathcal{R}}^\pi$ of r. Note that, if r does not have an immediate successor in $\hat{\mathcal{R}}^\pi$, then $S_r^\pi = S_{\min}^\pi$, and $p_r^\pi = L_r^\pi(q^\pi) = 1$ for any $q^\pi \in \Delta_{\hat{\mathcal{R}}^\pi}$, and thus $S_r^\pi \varphi^\pi(p_r^\pi) = S_{\min}^\pi \varphi^\pi(1)$ is a constant independent of q^π. Hence

$$\begin{aligned}
\phi(q) \;&=\; \sum_{\pi\in\Pi} D^\pi \sum_{r\in\hat{\mathcal{R}}^\pi} S_r^\pi \left[\varphi^\pi(p_r^\pi) - \varphi^\pi(p_{r-1}^\pi)\right] \\
&=\; \sum_{\pi\in\Pi} D^\pi \sum_{r\in\hat{\mathcal{R}}^\pi} \left[\mathbb{1}\{r+1 \in \hat{\mathcal{R}}^\pi\}\left(S_r^\pi - S_{r+1}^\pi\right)\phi_r^\pi(q^\pi) + \mathbb{1}\{r+1 \notin \hat{\mathcal{R}}^\pi\} S_{\min}^\pi \varphi^\pi(1)\right]
\end{aligned}$$

LEMMA 3. *The function $\phi : \prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi} \to \mathbb{R}$ is convex and continuous, and*

$$\partial\phi(q) \;=\; \prod_{\pi\in\Pi}\left(D^\pi \sum_{\{r\in\hat{\mathcal{R}}^\pi\,:\, r+1\in\hat{\mathcal{R}}^\pi\}} \left(S_r^\pi - S_{r+1}^\pi\right)\left(\mathbb{1}\left\{r' \preceq r\right\},\, r' \in \hat{\mathcal{R}}^\pi\right)(G^\pi)^{-1}\left(L_r^\pi(q^\pi)\right) + \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)\right)$$

Recall that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is assumed to be nondecreasing. Next we show that it is sufficient for ψ to be convex. For any $x > 0$, let $t_a^-(x) := \lim_{y\uparrow x} t_a(y)$ denote the left limit of t_a at x, and for any $x \geq 0$, let $t_a^+(x) := \lim_{y\downarrow x} t_a(y)$ denote the right limit of t_a at x. It follows from t_a being nondecreasing that $t_a^-(x) \leq t_a^+(x)$ for all $x > 0$.

LEMMA 4. *The function ψ is convex and continuous on $\prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi}$, and*

$$\partial\psi(q) \;=\; \left\{g = \left(g_r^\pi, \pi\in\Pi, r\in\hat{\mathcal{R}}^\pi\right) \,:\, g_r^\pi = D^\pi \sum_{a\in\mathcal{A}_r^\pi} \tau_a,\; \tau_a \in \left[t_a^-(x_a(q)), t_a^+(x_a(q))\right]\;\forall\, a\in\mathcal{A}\right\} + \prod_{\pi\in\Pi} \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$$

with $\left[t_a^-(0), t_a^+(0)\right] := \left(-\infty, t_a^+(0)\right]$.

To convert an optimal solution q of problem (18) to a pair (q, w) that satisfies the equilibrium conditions (12)–(15), we will assume that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, in addition to being nondecreasing. Then Lemma 4 simplifies to Lemma 5.

LEMMA 5. *If $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, then*

$$\partial\psi(q) \;=\; \left\{g = \left(g_r^\pi, \pi\in\Pi, r\in\hat{\mathcal{R}}^\pi\right) \,:\, g_r^\pi = D^\pi \sum_{a\in\mathcal{A}_r^\pi} t_a(x_a(q)) = D^\pi T_r^\pi(x(q))\right\} + \prod_{\pi\in\Pi} \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$$

4.2. Optimality Conditions

Next we establish necessary and sufficient conditions for $q^* \in \prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi}$ to be optimal for problem (18). For each $\pi \in \Pi$, let λ^π denote the dual variable associated with the simplex upper-bound constraint $\sum_{r\in\hat{\mathcal{R}}^\pi} q_r^\pi = 1$, and let $\lambda := (\lambda^\pi, \pi \in \Pi)$.

LEMMA 6. *A vector $q^* \in \prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi}$ is optimal for problem* (18) *if and only if there exist $\tau^* = (\tau_a^*, a \in \mathcal{A})$ with $\tau_a^* \in \left[t_a^-(x_a(q^*)), t_a^+(x_a(q^*))\right]$, $w^* = \left(w_r^{\pi*}, \pi \in \Pi, r \in \hat{\mathcal{R}}^\pi\right)$ with $w_r^{\pi*} \in (G^\pi)^{-1}\left(L_r^\pi(q^{\pi*})\right)$, and $\lambda^* = (\lambda^{\pi*}, \pi \in \Pi)$, such that the following holds:*

$$q_r^{\pi*}\left\{D^\pi\left[\sum_{a\in\mathcal{A}_r^\pi}\tau_a^* + \sum_{\{r'\in\hat{\mathcal{R}}^\pi : r\preceq r', r'+1\in\hat{\mathcal{R}}^\pi\}}\left(S_{r'}^\pi - S_{r'+1}^\pi\right)w_{r'}^{\pi*}\right] + \lambda^{\pi*}\right\} = 0 \qquad \forall\,\pi\in\Pi,\ \forall\, r\in\hat{\mathcal{R}}^\pi \tag{19}$$

$$D^\pi\left[\sum_{a\in\mathcal{A}_r^\pi}\tau_a^* + \sum_{\{r'\in\hat{\mathcal{R}}^\pi : r\preceq r', r'+1\in\hat{\mathcal{R}}^\pi\}}\left(S_{r'}^\pi - S_{r'+1}^\pi\right)w_{r'}^{\pi*}\right] + \lambda^{\pi*} \geq 0 \qquad \forall\,\pi\in\Pi,\ \forall\, r\in\hat{\mathcal{R}}^\pi \tag{20}$$

If $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous, in addition to being nondecreasing, then Lemma 6 simplifies to Lemma 7.

LEMMA 7. *Suppose that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous. Then, a vector $q^* \in \prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi}$ is optimal for problem* (18) *if and only if there exist $w^* = \left(w_r^{\pi*}, \pi \in \Pi, r \in \hat{\mathcal{R}}^\pi\right)$ with $w_r^{\pi*} \in (G^\pi)^{-1}\left(L_r^\pi(q^{\pi*})\right)$ and $\lambda^* = (\lambda^{\pi*}, \pi \in \Pi)$, such that the following holds:*

$$q_r^{\pi*}\left\{D^\pi\left[T_r^\pi(x(q^*)) + \sum_{\{r'\in\hat{\mathcal{R}}^\pi : r\preceq r', r'+1\in\hat{\mathcal{R}}^\pi\}}\left(S_{r'}^\pi - S_{r'+1}^\pi\right)w_{r'}^{\pi*}\right] + \lambda^{\pi*}\right\} = 0 \qquad \forall\,\pi\in\Pi,\ \forall\, r\in\hat{\mathcal{R}}^\pi \tag{21}$$

$$D^\pi\left[T_r^\pi(x(q^*)) + \sum_{\{r'\in\hat{\mathcal{R}}^\pi : r\preceq r', r'+1\in\hat{\mathcal{R}}^\pi\}}\left(S_{r'}^\pi - S_{r'+1}^\pi\right)w_{r'}^{\pi*}\right] + \lambda^{\pi*} \geq 0 \qquad \forall\,\pi\in\Pi,\ \forall\, r\in\hat{\mathcal{R}}^\pi \tag{22}$$

4.3. From Optimality to Equilibrium

Recall that optimization problem (18) has solution q, whereas the equilibrium conditions (12)–(15) are stated for a pair (q, w). Next we present Algorithm 1, that takes any subsets $\hat{\mathcal{R}}^\pi$, $\pi \in \Pi$, of paths, and optimal solution q for problem (18), as input, and produces the subset $\mathcal{R}_{\min}^\pi \subset \hat{\mathcal{R}}^\pi$ of paths on the efficient frontier, with thresholds $w = \left(w_r^\pi, \pi \in \Pi, r \in \{0\} \cup \hat{\mathcal{R}}^\pi\right)$, as output. Algorithm 1 makes provision for multiple tied paths, that is, paths r and r' with $S_r^\pi = S_{r'}^\pi$ and $T_r^\pi(x(q)) = T_{r'}^\pi(x(q))$, with zero or positive flow.

Algorithm 1 Threshold Selection Algorithm Thresholds$\big((\mathcal{N},\mathcal{A}),\Pi,\big((G^\pi,[\underline{w}^\pi,\overline{w}^\pi]),\pi\in\Pi\big),\big((t_a,s_a),a\in\mathcal{A}\big),\big(\hat{\mathcal{R}}^\pi,\pi\in\Pi\big),q\big)$

Require: Network $(\mathcal{N},\mathcal{A})$, set Π of OD pairs with cumulative distribution function G^π with support interval $[\underline{w}^\pi,\overline{w}^\pi]$ for each $\pi\in\Pi$, cost functions $t_a:\mathbb{R}_+\mapsto\mathbb{R}$ and costs $s_a\in\mathbb{R}_+$ for each $a\in\mathcal{A}$, any nonempty path sets $\hat{\mathcal{R}}^\pi$ indexed with $r=1,\dots,\hat{R}^\pi$ such that $S_1^\pi\geq\cdots\geq S_{\hat{R}^\pi}^\pi$ for each $\pi\in\Pi$, optimal solution $q:=(q_r^\pi,\pi\in\Pi,r=1,\dots,\hat{R}^\pi)\in\prod_{\pi\in\Pi}\Delta_{\hat{R}^\pi}$ for problem (18)

for $\pi\in\Pi$ **do**

$$r_1 \leftarrow \min\left\{\arg\min\left\{T_r^\pi(x(q))+\underline{w}^\pi S_r^\pi \,:\, r\in\hat{\mathcal{R}}^\pi\right\}\right\} \tag{23}$$

  for $r=0,\dots,r_1-1$ **do**
    $w_r^\pi\leftarrow\underline{w}^\pi$
  end for
  $\mathcal{R}_{\min}^\pi\leftarrow\{r_1\},\ \ell\leftarrow 1$
  while $r_\ell<\hat{R}^\pi$ **do**
    if $\left(S_{r_\ell}^\pi>S_{r_\ell+1}^\pi\right)$ **then**

$$w_\ell^{\min} \leftarrow \min\left\{\frac{T_r^\pi(x(q))-T_{r_\ell}^\pi(x(q))}{S_{r_\ell}^\pi-S_r^\pi} \,:\, r\in\{r_\ell+1,\dots,\hat{R}^\pi\}\right\},\qquad w_{r_\ell}^\pi \leftarrow \min\left\{w_\ell^{\min},\overline{w}^\pi\right\} \tag{24}$$

$$r_{\ell+1} \leftarrow \min\left\{\arg\min\left\{\frac{T_r^\pi(x(q))-T_{r_\ell}^\pi(x(q))}{S_{r_\ell}^\pi-S_r^\pi} \,:\, r\in\{r_\ell+1,\dots,\hat{R}^\pi\}\right\}\right\} \tag{25}$$

    else $\left(S_{r_\ell}^\pi=S_{r_\ell+1}^\pi\right)$

$$T_\ell^{\min} \leftarrow \min\left\{T_r^\pi(x(q)) \,:\, S_r^\pi=S_{r_\ell}^\pi,\ r\in\{r_\ell+1,\dots,\hat{R}^\pi\}\right\}$$

      if $\left(T_\ell^{\min}=T_{r_\ell}^\pi(x(q))\right)$ **then**

$$w_\ell^{\min} \leftarrow w_{r_\ell}^\pi \leftarrow \max\left\{w_{r_\ell-1}^\pi,\min\left\{w\in\left(G^\pi\right)^{-1}\left(L_{r_\ell}^\pi(q^\pi)\right)\right\}\right\} \tag{26}$$

$$r_{\ell+1} \leftarrow \min\left\{\arg\min\left\{T_r^\pi(x(q)) \,:\, S_r^\pi=S_{r_\ell}^\pi,\ r\in\{r_\ell+1,\dots,\hat{R}^\pi\}\right\}\right\} \tag{27}$$

      else $\left(T_\ell^{\min}>T_{r_\ell}^\pi(x(q))\right)$
        if $\left(\left\{r\in\hat{\mathcal{R}}^\pi \,:\, S_{r_\ell}^\pi>S_r^\pi\right\}\neq\varnothing\right)$ **then**

$$w_\ell^{\min} \leftarrow \min\left\{\frac{T_r^\pi(x(q))-T_{r_\ell}^\pi(x(q))}{S_{r_\ell}^\pi-S_r^\pi} \,:\, S_{r_\ell}^\pi>S_r^\pi,\ r\in\hat{\mathcal{R}}^\pi\right\},\qquad w_{r_\ell}^\pi \leftarrow \min\left\{w_\ell^{\min},\overline{w}^\pi\right\} \tag{28}$$

$$r_{\ell+1} \leftarrow \min\left\{\arg\min\left\{\frac{T_r^\pi(x(q))-T_{r_\ell}^\pi(x(q))}{S_{r_\ell}^\pi-S_r^\pi} \,:\, S_{r_\ell}^\pi>S_r^\pi,\ r\in\hat{\mathcal{R}}^\pi\right\}\right\} \tag{29}$$

        else $\left(\left\{r\in\hat{\mathcal{R}}^\pi \,:\, S_{r_\ell}^\pi>S_r^\pi\right\}=\varnothing\right)$

$$w_\ell^{\min} \leftarrow \overline{w}^\pi+1,\qquad w_{r_\ell}^\pi \leftarrow \overline{w}^\pi \tag{30}$$

        end if
      end if
    end if
    if $w_\ell^{\min}\leq\overline{w}^\pi$ **then**
      $\mathcal{R}_{\min}^\pi\leftarrow\mathcal{R}_{\min}^\pi\cup\{r_{\ell+1}\}$
    else
      $r_{\ell+1}\leftarrow\hat{R}^\pi$
    end if
    for $r=r_\ell+1,\dots,r_{\ell+1}-1$ **do**
      $w_r^\pi\leftarrow w_{r_\ell}^\pi$
    end for
    $\ell\leftarrow\ell+1$
  end while

$$w_{\hat{R}^\pi}^\pi \leftarrow \overline{w}^\pi \tag{31}$$

end for

Output $w=\left(w_r^\pi,\pi\in\Pi,r\in\{0\}\cup\hat{\mathcal{R}}^\pi\right)$ and $\mathcal{R}_{\min}^\pi,\pi\in\Pi$

A brief summary of the steps of Algorithm 1 is given next. First, for each OD pair π, the algorithm selects the path with the least cost $C_r^\pi(\underline{w}^\pi, x(q))$, initializes $w_r^\pi \leftarrow \underline{w}^\pi$ for lower indexed (inefficient) paths, and starts the list $\mathcal{R}_{\min}^\pi$ of efficient paths. Second, each iteration applies one of the cases (24), (26), (28), or (30) to determine the next threshold value and the next efficient path. Third, whenever the next threshold value is feasible, the algorithm appends the next efficient path to the list $\mathcal{R}_{\min}^\pi$, and fills in the threshold values for inefficient paths with intermediate indices, while preserving monotonicity of the threshold sequence. Finally, it assigns terminal inefficient paths threshold values of $\overline{w}^\pi$, and outputs both the thresholds w and the lists $\mathcal{R}_{\min}^\pi$ of efficient paths.

EXAMPLE 4. In this example, we apply Algorithm 1 to the path data of Example 1 and the path fractions q of Example 3. Example 6 will verify that these fractions q solve problem (18).

At $\underline{w}^\pi = 0$, relation (23) selects path r_1. The minimum in (24) is 0.01, attained by both paths r_2 and r_3, so the minimum-index rule in (25) selects r_2 and sets $w_1^\pi = \min\{0.01, 0.14\}$. Because $S_{r_2}^\pi = S_{r_3}^\pi$, $T_{r_2}^\pi(x(q)) = T_{r_3}^\pi(x(q))$, and $L_{r_2}^\pi(q^\pi) = 0.4$, case (26) gives $w_2^\pi = \max\left\{0.01, \min\left\{w \in (G^\pi)^{-1}(0.4)\right\}\right\} = 0.03$, and relation (27) selects r_3.

Starting from r_3, the minimum in (24) is 0.07, attained by r_6 and r_9. The minimum-index rule selects r_6, sets $w_3^\pi = \min\{0.07, 0.14\}$, and the fill step sets $w_4^\pi = w_5^\pi = 0.07$. Next, starting from r_6, the minimum in (24) is also 0.07, attained by r_9, setting $w_6^\pi = \min\{0.07, 0.14\}$, and the fill step sets $w_7^\pi = w_8^\pi = 0.07$. Starting from r_9, the minimum in (24) is 0.12, attained by r_{10}, setting $w_9^\pi = \min\{0.12, 0.14\}$. Finally, the terminal step sets $w_{10}^\pi = \overline{w}^\pi = 0.14$. Consequently,

$$\mathcal{R}_{\min}^\pi = \{r_1, r_2, r_3, r_6, r_9, r_{10}\}, \qquad (w_0^\pi, \ldots, w_{10}^\pi) = (0, .01, .03, .07, .07, .07, .07, .07, .07, .12, .14) \tag{32}$$

which is the threshold vector of Example 3. Figure 3 verifies that (15) holds, and Figure 4 shows the corresponding optimality intervals. □

In the remainder of the paper we assume that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous. Example 5 shows that continuity of t_a is needed to guarantee existence of an equilibrium.

EXAMPLE 5. Consider a network with a single OD pair π with demand $D^\pi = 1$, and two disjoint paths, indexed 1 and 2. The weight w is 0 with probability 1, that is, it is a classical single-criterion traffic assignment problem. The travel time on path 1 is $t_1(x_1) = 1 + 2\mathbb{1}\{x_1 \geq 1/2\}$, and the travel time on path 2 is $t_2(x_2) = 2$. Note that $x_1(q) = q_1$ and $x_2(q) = q_2$. Any solution with $q_1 < 1/2$ satisfies $q_2 > 1/2$ but $C_1^\pi(w, x(q)) < C_2^\pi(w, x(q))$, and any solution with $q_1 \geq 1/2$ satisfies $C_1^\pi(w, x(q)) > C_2^\pi(w, x(q))$. Thus there is no solution that satisfies equilibrium condition (14). □

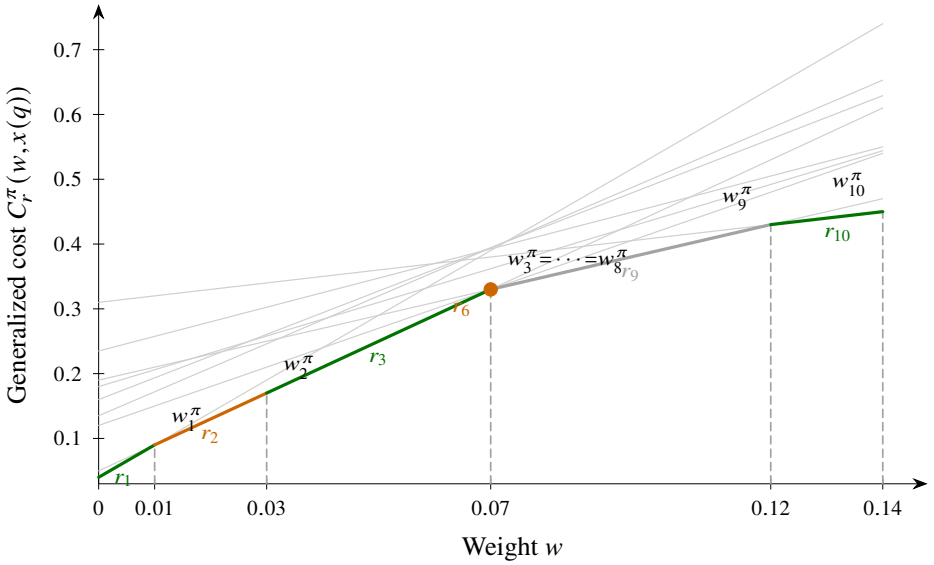


Figure 4 Application of Threshold Algorithm 1 to the data of Examples 1 and 3. The thin gray lines are the distinct bicriteria path-cost functions $C_r^\pi(w,x)$, $r \in \hat{\mathcal{R}}^\pi$ (note that the functions for r_2 and r_3 coincide), and the thick segments show the intervals assigned along the lower envelope. The values 0.01, 0.07, and 0.12 are intersection thresholds where the slope of the lower envelope changes. The value $w_2^\pi = 0.03$ is not a kink, rather it is the quantile threshold that separates the intervals assigned to the tied paths r_2 and r_3. The orange point at $w = 0.07$ is the singleton interval assigned to r_6, and the gray segment is the positive-length optimality interval of the zero-flow path r_9.

LEMMA 8. *Assume that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous for all $a \in \mathcal{A}$. Consider any optimal solution q for problem* (18)*, and the resulting output w and $\mathcal{R}^\pi_{\min}$, $\pi \in \Pi$, of Algorithm 1. Then the following hold:*

(1) The thresholds w satisfy $\underline{w}^\pi = w_0^\pi \le w_1^\pi \le \cdots \le w_{\hat{R}^\pi}^\pi = \overline{w}^\pi$ for all $\pi \in \Pi$.

(2) Every path with positive flow is an efficient path, that is, $\left\{r \in \hat{\mathcal{R}}^\pi : q_r^\pi > 0\right\} \subset \mathcal{R}^\pi_{\min}$.

(3) The thresholds w satisfy $w_r^\pi \in (G^\pi)^{-1}\left(L_r^\pi(q^\pi)\right)$ for all $\pi \in \Pi$, $r \in \hat{\mathcal{R}}^\pi$.

(4) For every $\pi \in \Pi$ and every $r_\ell \in \mathcal{R}^\pi_{\min}$ it holds that

$$T^\pi_{r_\ell}(x(q)) + wS^\pi_{r_\ell} \;\le\; T^\pi_r(x(q)) + wS^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi,\ \forall\, w \in [w^\pi_{r_{\ell-1}}, w^\pi_{r_\ell}] \tag{33}$$

where $w^\pi_{r_0} = w_0^\pi = \underline{w}^\pi$.

THEOREM 1. *Assume that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous for all $a \in \mathcal{A}$. Consider any optimal solution q for problem* (18) *with $\hat{\mathcal{R}}^\pi = \mathcal{R}^\pi$ for all $\pi \in \Pi$, and the resulting output w of Algorithm 1. Then (q, w) satisfies the equilibrium conditions* (12)–(15).

4.4. Solving the Optimization Problem

Section 4.3 showed how an optimal solution q of optimization problem (18) can be converted to an equilibrium (q, w) that satisfies conditions (12)–(15). In this section we propose an algorithm to solve optimization problem (18) for the full set $\hat{\mathcal{R}}^\pi = \mathcal{R}^\pi$, $\pi \in \Pi$, of OD paths. Lemma 9 shows that the optimality conditions (21)–(22) are satisfied by (q, λ, w) for any optimal solution q of problem (18), λ given by (34), and with w given by Algorithm 1.

LEMMA 9. *Assume that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous for all $a \in \mathcal{A}$. Consider any optimal solution q for problem* (18)*, and the resulting output w and $\mathcal{R}^\pi_{\min} = \{r_1, \ldots, r_L\}$ of Algorithm 1. Let $\hat{\mathcal{R}}^\pi$ be indexed with $r = 1, \ldots, \hat{R}^\pi$, and let*

$$\lambda^\pi \;:=\; -D^\pi \left[T^\pi_{r_L}(x(q)) + \overline{w}^\pi \left(S^\pi_{r_L} - S^\pi_{\hat{R}^\pi} \right) \right] \qquad \forall\, \pi \in \Pi \tag{34}$$

Then (q, λ, w) satisfies the optimality conditions (21)–(22)*.*

Problem (18) is a convex optimization problem, and therefore can be solved by general convex optimization algorithms. However, typical instances of the traffic assignment problem have many paths for most OD pairs, only a small fraction of which have positive flow in an equilibrium. Thus, we propose Algorithm BTA (Algorithm 2) that exploits both convexity as well as the sparsity of typical solutions. More specifically, Algorithm BTA maintains a (typically small) subset $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$ of paths for each OD pair $\pi \in \Pi$, solves the convex problem (18) for the subsets $\hat{\mathcal{R}}^\pi$, $\pi \in \Pi$, to produce flow fractions $\hat{q} = \left(\hat{q}^\pi_r, \pi \in \Pi, r \in \hat{\mathcal{R}}^\pi \right)$, update arc flows $x_a(\hat{q})$ and travel times $t_a(x_a(\hat{q}))$, $a \in \mathcal{A}$, and then solves a parametric shortest path problem for each origin $u \in \mathcal{N}$ such that $\mathcal{V}_u := \{v \in \mathcal{N} : (u, v) \in \Pi\} \neq \emptyset$. Algorithms for the parametric shortest path problem have been proposed in Karp and Orlin (1981). Algorithm 3 solves the parametric shortest path problem for a given origin $u \in \mathcal{N}$. If the parametric shortest path problems produce paths not yet in the subsets $\hat{\mathcal{R}}^\pi$, then the new paths are added to $\hat{\mathcal{R}}^\pi$, and Algorithm BTA returns to solving problem (18) for the updated subsets $\hat{\mathcal{R}}^\pi$, $\pi \in \Pi$; otherwise Algorithm BTA returns $\hat{q} = \left(\hat{q}^\pi_r, \pi \in \Pi, r \in \hat{\mathcal{R}}^\pi \right)$. Note that $\hat{q}$ contains path flow fractions $\hat{q}^\pi_r$ for paths $r \in \hat{\mathcal{R}}^\pi$ only. Lemma 10 shows that the expanded solution q^* with $q^{\pi *}_r = \hat{q}^\pi_r$ for $r \in \hat{\mathcal{R}}^\pi$ and $q^{\pi *}_r = 0$ for $r \in \mathcal{R}^\pi \setminus \hat{\mathcal{R}}^\pi$ is optimal for problem (18) with the complete set of paths. As shown in Section 4.3, Algorithm 1 can be used to convert q^* to an equilibrium (q^*, w^*) with the complete set of paths.

REMARK 1. Algorithm 3 returns both shortest paths $\mathcal{R}^{(u,v)+}$ as well as thresholds $w^{(u,v)+}_0, \ldots, w^{(u,v)+}_{R^{(u,v)+}}$. The thresholds are not used in Algorithm BTA, but facilitate the following

Algorithm 2 Bicriteria Traffic Assignment Algorithm $\text{BTA}\big((\mathcal{N},\mathcal{A}),\Pi,\big((D^\pi,G^\pi,[\underline{w}^\pi,\overline{w}^\pi]),\pi\in\Pi\big),\big((t_a,s_a),a\in\mathcal{A}\big)\big)$

Require: Network $(\mathcal{N},\mathcal{A})$, set Π of OD pairs with demand D^π and cumulative distribution function G^π with support $[\underline{w}^\pi,\overline{w}^\pi]$ for each $\pi\in\Pi$, cost functions $t_a:\mathbb{R}_+\mapsto\mathbb{R}_+$ and costs $s_a\in\mathbb{R}_+$ for each $a\in\mathcal{A}$

for $u\in\mathcal{N}$ **do**
  $\mathcal{V}_u\leftarrow\{v\in\mathcal{N}:(u,v)\in\Pi\}$
  if $\mathcal{V}_u\neq\emptyset$ **then**
    $\big((\hat{\mathcal{R}}^{(u,v)},w^{(u,v)}),v\in\mathcal{V}_u\big)\leftarrow\text{BPSP}\big((\mathcal{N},\mathcal{A}),u,\mathcal{V}_u,((t_a(0),s_a),a\in\mathcal{A}),([\underline{w}^{(u,v)},\overline{w}^{(u,v)}],v\in\mathcal{V}_u)\big)$
  end if
end for
$\eta\leftarrow 1$
while $\eta=1$ **do**
  $\eta\leftarrow 0$
  $\hat{q}\leftarrow\arg\min\big\{\Upsilon(q):q\in\prod_{\pi\in\Pi}\Delta_{\hat{\mathcal{R}}^\pi}\big\}$
  for $u\in\mathcal{N}$ such that $\mathcal{V}_u\neq\emptyset$ **do**
    $\big((\mathcal{R}^{(u,v)+},w^{(u,v)+}),v\in\mathcal{V}_u\big)\leftarrow\text{BPSP}\big((\mathcal{N},\mathcal{A}),u,\mathcal{V}_u,((t_a(x_a(\hat{q})),s_a),a\in\mathcal{A}),([\underline{w}^{(u,v)},\overline{w}^{(u,v)}],v\in\mathcal{V}_u)\big)$
    for $v\in\mathcal{V}_u$ **do**
      if $\mathcal{R}^{(u,v)+}\not\subset\hat{\mathcal{R}}^{(u,v)}$ **then**
        $\hat{\mathcal{R}}^{(u,v)}\leftarrow\hat{\mathcal{R}}^{(u,v)}\cup\mathcal{R}^{(u,v)+}$
        $\eta\leftarrow 1$
      end if
    end for
  end for
end while
Output $\hat{q}$ and $\hat{\mathcal{R}}^\pi$ for all $\pi\in\Pi$

point. Lemma 8 established that, given any optimal solution q for problem (18), the thresholds produced by Algorithm 1 satisfy (13), (14), and (15). In contrast, as shown next, the thresholds produced by Algorithm 3 satisfy (13) and (14), but not (15). Consider the data of Example 3, and suppose that each path r_i is associated with a separate arc also denoted with r_i and that the input data with $t_{(i,j)}=t_{(i,j)}(x_{(i,j)}(q))$ is provided to Algorithm 3. Then $w_0^\pi=\underline{w}=0$, and path r_1 is the unique shortest path with arc costs $c_{(i,j)}=t_{(i,j)}+\underline{w}s_{(i,j)}$. Then $w'=0.01$, $\mathcal{A}'=\{r_2,r_3\}$, and $w_1^\pi=w'=0.01$. Algorithm 3 does not specify which arc in $\mathcal{A}'$ is selected to construct the next tree. Whichever arc in $\mathcal{A}'$ is selected, in the next iteration $w'=0.07$, $\mathcal{A}'=\{r_6,r_9\}$, and $w_2^\pi=w'=0.07$. Recall from Example 4 that $w_2^\pi=(G^\pi)^{-1}(L_{r_2}^\pi(q^\pi))=0.03$, which cannot be obtained by solving the parametric shortest path problem. The point is that although a parametric shortest path algorithm computes a sequence of thresholds, when there are ties on the efficient frontier, these thresholds may not satisfy (15), because the distribution G^π is immaterial to the parametric shortest path problem. Algorithm 1 is designed to produce thresholds that satisfy both the quantile condition (15), as well as, similar to a parametric shortest path algorithm, the minimal cost condition (14).

LEMMA 10. *Assume that $t_a:\mathbb{R}_+\mapsto\mathbb{R}_+$ is continuous for all $a\in\mathcal{A}$. For each $\pi\in\Pi$, let $\mathcal{R}^\pi$ denote the complete path set and let $\hat{\mathcal{R}}^\pi\subset\mathcal{R}^\pi$. Consider any optimal solution $\hat{q}$ for problem* (18) *with path sets $\hat{\mathcal{R}}^\pi$, $\pi\in\Pi$. Let $\hat{w}=\left(\hat{w}_r^\pi,\pi\in\Pi,r\in\{0\}\cup\hat{\mathcal{R}}^\pi\right)$ and $\hat{\mathcal{R}}_{\min}^\pi=\{\hat{r}_1,\ldots,\hat{r}_{\hat{L}}\}$ denote the output of Algorithm 1 with input $\hat{\mathcal{R}}^\pi$ and $\hat{q}$.*

Algorithm 3 Bicriteria Parametric Shortest Path Algorithm from One Origin to All Destinations $\mathrm{BPSP}\big((\mathcal{N},\mathcal{A}),u,\mathcal{V}_u,((t_a,s_a),a\in\mathcal{A}),([\underline{w}^{(u,v)},\overline{w}^{(u,v)}],v\in\mathcal{V}_u)\big)$

Require: Network $(\mathcal{N},\mathcal{A})$, origin u, set $\mathcal{V}_u$ of destinations, costs $t_{(i,j)}\in\mathbb{R}_+$ and $s_{(i,j)}\in\mathbb{R}_+$ for each $(i,j)\in\mathcal{A}$, support intervals $[\underline{w}^{(u,v)},\overline{w}^{(u,v)}]$ for all $v\in\mathcal{V}_u$

$\underline{w}_u \leftarrow \min\left\{\underline{w}^{(u,v)} : v\in\mathcal{V}_u\right\}$ and $\overline{w}_u \leftarrow \max\left\{\overline{w}^{(u,v)} : v\in\mathcal{V}_u\right\}$
$w \leftarrow w' \leftarrow \underline{w}_u$
Compute a shortest path tree $(\mathcal{N},\mathcal{A}_u)$ from origin u to destinations $v\in\mathcal{V}_u$ on network $(\mathcal{N},\mathcal{A})$ with arc costs $c_{(i,j)} = t_{(i,j)} + w s_{(i,j)}$
for $i\in\mathcal{N}$ **do**
    $T_i \leftarrow$ the t-cost from u to i in the shortest path tree $(\mathcal{N},\mathcal{A}_u)$
    $S_i \leftarrow$ the s-cost from u to i in the shortest path tree $(\mathcal{N},\mathcal{A}_u)$
end for
for $v\in\mathcal{V}_u$ **do**
    if $\underline{w}^{(u,v)} \le w$ **then**
        $\mathcal{R}^{(u,v)+} \leftarrow \{$path from u to v in the shortest path tree $(\mathcal{N},\mathcal{A}_u)\}$
        $R^{(u,v)+} \leftarrow 1$
    else
        $\mathcal{R}^{(u,v)+} \leftarrow \varnothing$
        $R^{(u,v)+} \leftarrow 0$
    end if
    $w_0^{(u,v)+} \leftarrow \underline{w}^{(u,v)}$
end for
while $w' \le \overline{w}_u$ and $\left\{(i,j)\in\mathcal{A}\setminus\mathcal{A}_u : S_i + s_{(i,j)} < S_j\right\} \neq \varnothing$ **do**

$$w' \leftarrow \min\left\{\frac{T_i + t_{(i,j)} - T_j}{S_j - S_i - s_{(i,j)}} : (i,j)\in\mathcal{A}\setminus\mathcal{A}_u,\ S_i + s_{(i,j)} < S_j\right\}$$
$$\mathcal{A}' \leftarrow \arg\min\left\{\frac{T_i + t_{(i,j)} - T_j}{S_j - S_i - s_{(i,j)}} : (i,j)\in\mathcal{A}\setminus\mathcal{A}_u,\ S_i + s_{(i,j)} < S_j\right\}$$

    if $w' \le \overline{w}_u$ **then**
        Select $(i,j)\in\mathcal{A}'$
        Find $(i',j)\in\mathcal{A}_u$
        $\mathcal{A}'_u \leftarrow \mathcal{A}_u \cup \{(i,j)\}\setminus\{(i',j)\}$
        $\mathcal{N}' \leftarrow \{j\}$
        $\Delta T \leftarrow T_i + t_{(i,j)} - T_j$
        $\Delta S \leftarrow S_i + s_{(i,j)} - S_j$
        while $\mathcal{N}' \neq \varnothing$ **do**
            Remove $i''\in\mathcal{N}'$
            Update $T_{i''} \leftarrow T_{i''} + \Delta T$ and $S_{i''} \leftarrow S_{i''} + \Delta S$
            if $i''\in\mathcal{V}_u$ **then**
                Update $(\mathcal{R}^{(u,i'')+}, w^{(u,i'')+}) \leftarrow \mathrm{UpdatePSP}\big((\mathcal{N},\mathcal{A}_u),(\mathcal{N},\mathcal{A}'_u),u,i'',[\underline{w}^{(u,i'')},\overline{w}^{(u,i'')}],w,w',\mathcal{R}^{(u,i'')+},w^{(u,i'')+}\big)$
            end if
            for $(i'',j')\in\mathcal{A}_u$ **do**
                $\mathcal{N}' \leftarrow \mathcal{N}'\cup\{j'\}$
            end for
        end while
        $w \leftarrow w'$
        $\mathcal{A}_u \leftarrow \mathcal{A}'_u$
    end if
end while
for $v\in\mathcal{V}_u$ **do**
    if $w < \underline{w}^{(u,v)}$ **then**
        $\mathcal{R}^{(u,v)+} \leftarrow \{$path from u to v in the tree $(\mathcal{N},\mathcal{A}_u)\}$
        $w_1^{(u,v)+} \leftarrow \overline{w}^{(u,v)}$
        $R^{(u,v)+} \leftarrow 1$
    else $(\underline{w}^{(u,v)} \le w)$
        $w_{R^{(u,v)+}}^{(u,v)+} \leftarrow \overline{w}^{(u,v)}$
    end if
end for
Output $\mathcal{R}^{(u,v)+}$ and $w_0^{(u,v)+},\ldots,w_{R^{(u,v)+}}^{(u,v)+}$ for all $v\in\mathcal{V}_u$

Algorithm 4 Update Parametric Shortest Paths and Thresholds UpdatePSP$\big((\mathcal{N},\mathcal{A}_u),(\mathcal{N},\mathcal{A}'_u),u,v,[\underline{w}^{(u,v)},\overline{w}^{(u,v)}],w,w',\mathcal{R}^{(u,v)+},w^{(u,v)+}\big)$

Require: Trees $(\mathcal{N},\mathcal{A}_u)$ and $(\mathcal{N},\mathcal{A}'_u)$, origin u, destination v, support interval $[\underline{w}^{(u,v)},\overline{w}^{(u,v)}]$, current thresholds $w \le w'$, path list $\mathcal{R}^{(u,v)+}$ with $|\mathcal{R}^{(u,v)+}| = R^{(u,v)+}$, threshold list $w^{(u,v)+}$

if $w < \underline{w}^{(u,v)} \le \overline{w}^{(u,v)} < w'$ **then**
  $\mathcal{R}^{(u,v)+} \leftarrow \{$path from u to v in the tree $(\mathcal{N},\mathcal{A}_u)\}$
  $w^{(u,v)+}_1 \leftarrow \overline{w}^{(u,v)}$
  $R^{(u,v)+} \leftarrow 1$
else if $w < \underline{w}^{(u,v)} \le w' \le \overline{w}^{(u,v)}$ **then**
  $\mathcal{R}^{(u,v)+} \leftarrow \{$path from u to v in the tree $(\mathcal{N},\mathcal{A}_u)\}$
  $w^{(u,v)+}_1 \leftarrow w'$
  $\mathcal{R}^{(u,v)+} \leftarrow \mathcal{R}^{(u,v)+} \cup \{$path from u to v in the tree $(\mathcal{N},\mathcal{A}'_u)\}$
  $R^{(u,v)+} \leftarrow 2$
else if $\underline{w}^{(u,v)} \le w \le \overline{w}^{(u,v)} < w'$ **then**
  $w^{(u,v)+}_{R^{(u,v)+}} \leftarrow \overline{w}^{(u,v)}$
else if $\underline{w}^{(u,v)} \le w \le w' \le \overline{w}^{(u,v)}$ **then**
  $w^{(u,v)+}_{R^{(u,v)+}} \leftarrow w'$
  $\mathcal{R}^{(u,v)+} \leftarrow \mathcal{R}^{(u,v)+} \cup \{$path from u to v in the tree $(\mathcal{N},\mathcal{A}'_u)\}$
  $R^{(u,v)+} \leftarrow R^{(u,v)+} + 1$
end if
Output path list $\mathcal{R}^{(u,v)+}$, threshold list $w^{(u,v)+}$

Let $q^* = \left(q^{\pi *}_r, \pi \in \Pi, r \in \mathcal{R}^\pi\right)$ *be as follows: For each* $r \in \hat{\mathcal{R}}^\pi$ *let* $q^{\pi *}_r = \hat{q}^\pi_r$*, and for each* $r \in \mathcal{R}^\pi \setminus \hat{\mathcal{R}}^\pi$ *let* $q^{\pi *}_r = 0$*. Then* $x(q^*) = x(\hat{q})$*.*

Let $w^* = \left(w^{\pi *}_r, \pi \in \Pi, r \in \{0\} \cup \mathcal{R}^\pi\right)$ *and* $\mathcal{R}^{\pi *}_{\min} = \{r^*_1, \ldots, r^*_{L^*}\}$ *denote the output of Algorithm 1 with input* $\mathcal{R}^\pi$ *and* q^**. Let* $\mathcal{R}^\pi$ *be indexed with* $r = 1, \ldots, R^\pi$*, and let*

$$\lambda^{\pi *} \;:=\; -D^\pi \left[T^\pi_{r^*_{L^*}}(x(q^*)) + \overline{w}^\pi \left(S^\pi_{r^*_{L^*}} - S^\pi_{R^\pi} \right) \right] \qquad \forall\, \pi \in \Pi \tag{35}$$

Let $\lambda^* := (\lambda^{\pi *}, \pi \in \Pi)$*. For each* $\pi \in \Pi$*, let* $\mathcal{R}^{\pi +}$ *denote an optimal set of paths for the parametric shortest path problem (over the complete path set* $\mathcal{R}^\pi$*) with arc costs* $t_a(x_a(\hat{q}))$ *and* s_a*. That is,* $\mathcal{R}^{\pi +}$ *can be indexed such that* $\mathcal{R}^{\pi +} = \{r^1, \ldots, r^k\}$ *and there are thresholds* $\underline{w}^\pi = w^\pi_{r^0} \le w^\pi_{r^1} \le \cdots \le w^\pi_{r^{k-1}} \le w^\pi_{r^k} = \overline{w}^\pi$ *such that for all* $i = 1, \ldots, k$ *it holds that* $C^\pi_{r^i}(w, x(\hat{q})) \le C^\pi_{r'}(w, x(\hat{q}))$ *for all* $r' \in \mathcal{R}^\pi$ *and* $w \in [w^\pi_{r^{i-1}}, w^\pi_{r^i}]$*. Suppose that* $\mathcal{R}^{\pi +} \subset \hat{\mathcal{R}}^\pi$ *for each* $\pi \in \Pi$*. Then it holds that* (q^*, λ^*, w^*) *satisfies the optimality conditions for the complete path set:*

$$q^{\pi *}_r \left\{ D^\pi \left[T^\pi_r(x(q^*)) + \sum_{r'=r}^{R^\pi - 1} \left(S^\pi_{r'} - S^\pi_{r'+1} \right) w^{\pi *}_{r'} \right] + \lambda^{\pi *} \right\} \;=\; 0 \qquad \forall\, \pi \in \Pi,\ \forall\, r \in \mathcal{R}^\pi \tag{36}$$

$$D^\pi \left[T^\pi_r(x(q^*)) + \sum_{r'=r}^{R^\pi - 1} \left(S^\pi_{r'} - S^\pi_{r'+1} \right) w^{\pi *}_{r'} \right] + \lambda^{\pi *} \;\ge\; 0 \qquad \forall\, \pi \in \Pi,\ \forall\, r \in \mathcal{R}^\pi \tag{37}$$

THEOREM 2. *Algorithm BTA returns a solution* $\hat{q} = (\hat{q}^\pi_r, \pi \in \Pi, r \in \hat{\mathcal{R}}^\pi)$ *in* $O\left(\sum_{\pi \in \Pi} \left|\hat{\mathcal{R}}^\pi\right|\right)$ *steps. Then* $q^* = (q^{\pi *}_r, \pi \in \Pi, r \in \mathcal{R}^\pi)$*, where* $q^{\pi *}_r = \hat{q}^\pi_r$ *for all* $\pi \in \Pi$*,* $r \in \hat{\mathcal{R}}^\pi$ *and* $q^{\pi *}_r = 0$ *for all* $\pi \in \Pi$*,*

$r \in \mathcal{R}^\pi \setminus \hat{\mathcal{R}}^\pi$, is optimal for problem (18) *with the complete path set. Given q^* and the resulting output w^* of Algorithm 1, (q^*, w^*) is an equilibrium of the BiTAP.*

EXAMPLE 6. We present a network for which (q, w) given in Examples 1 and 3 is an equilibrium. We also present the steps of Algorithm BTA. Let

$$\mathcal{N} = \{u, v, z_1, \ldots, z_{10}\}, \qquad \mathcal{A} = \{(u, z_i), (z_i, v) : i = 1, \ldots, 10\},$$

let $\Pi = \{\pi\}$ with $\pi = (u, v)$, let $D^\pi = 1$, and let path $r_i = ((u, z_i), (z_i, v))$. For every i, let $t_{(u,z_i)}(y) = 0$, $s_{(u,z_i)} = 0$, and $s_{(z_i,v)} = S^\pi_{r_i}$, with S^π as in Example 1, and let

$$\begin{aligned}
&t_{(z_1,v)}(y) = 0.03 + 0.1y, \quad && t_{(z_2,v)}(y) = 0.05,\\
&t_{(z_3,v)}(y) = 0.04 + y/30, \quad && t_{(z_4,v)}(y) = 0.135,\\
&t_{(z_5,v)}(y) = 0.16, && t_{(z_6,v)}(y) = 0.11 + 0.1y,\\
&t_{(z_7,v)}(y) = 0.18, && t_{(z_8,v)}(y) = 0.235,\\
&t_{(z_9,v)}(y) = 0.19, && t_{(z_{10},v)}(y) = 0.31.
\end{aligned} \tag{38}$$

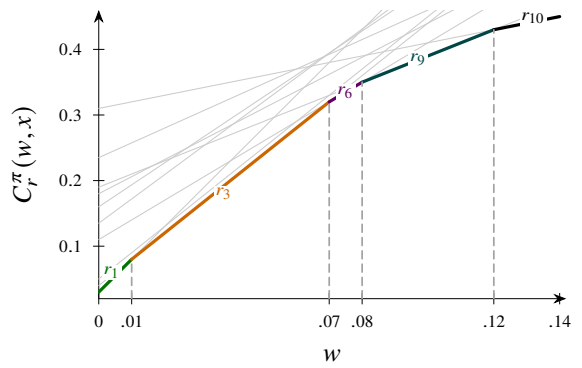
The paths are arc-disjoint, so (11) and (38) give

$$q^* = (.1, .3, .3, 0, 0, .1, 0, 0, 0, .2) \quad \Rightarrow \quad T^\pi(x(q^*)) = (.04, .05, .05, .135, .16, .12, .18, .235, .19, .31), \tag{39}$$

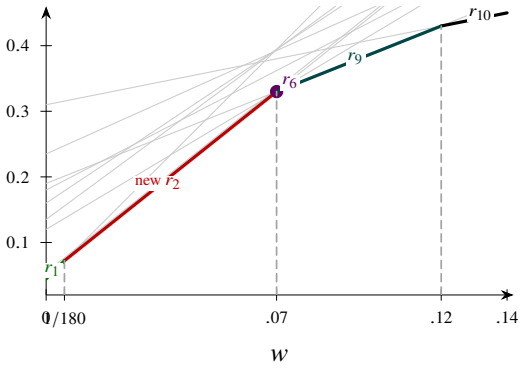
as required. Starting with arc travel times $t_a(0)$, Algorithm 3 gives initial path set $\hat{\mathcal{R}}^\pi = \mathcal{R}^{\pi+} = \{r_1, r_3, r_6, r_9, r_{10}\}$. Path r_2 is omitted because $S^\pi_{r_2} = S^\pi_{r_3}$ and $t_{(z_2,v)}(0) = .05 > .04 = t_{(z_3,v)}(0)$; Figure 5(a) shows that the other omitted paths also lie above the efficient frontier.

The optimal solution $\hat{q}$ for problem (18) with initial path set $\hat{\mathcal{R}}^\pi$ is $(\hat{q}^\pi_{r_1}, \hat{q}^\pi_{r_3}, \hat{q}^\pi_{r_6}, \hat{q}^\pi_{r_9}, \hat{q}^\pi_{r_{10}}) = \left(\frac{13}{90}, \frac{13}{30}, \frac{1}{10}, \frac{11}{90}, \frac{1}{5}\right)$, with thresholds $(\hat{w}^\pi_{r_1}, \hat{w}^\pi_{r_3}, \hat{w}^\pi_{r_6}, \hat{w}^\pi_{r_9}, \hat{w}^\pi_{r_{10}}) = \left(\frac{1}{100}, \frac{59}{900}, \frac{7}{100}, \frac{3}{25}, \frac{7}{50}\right)$. Optimality conditions (21)–(22) hold with $\lambda^\pi = -31/100$.

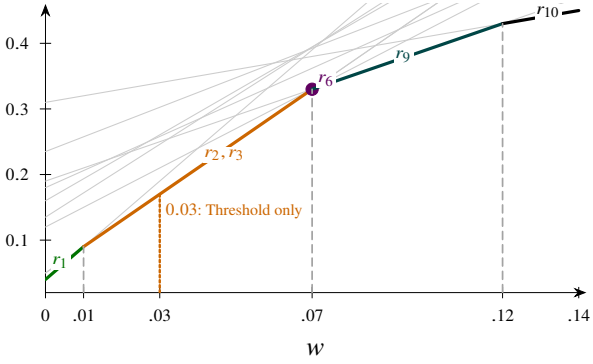
At this solution, $T^\pi_{r_2}(x(\hat{q})) = 1/20 < 49/900 = T^\pi_{r_3}(x(\hat{q}))$. Algorithm 3 with arc travel times $t_a(x_a(\hat{q}))$ gives path set $\mathcal{R}^{\pi+} = \{r_1, r_2, r_6, r_9, r_{10}\}$, and thus $\hat{\mathcal{R}}^\pi \leftarrow \hat{\mathcal{R}}^\pi \cup \{r_2\}$. The optimal solution for problem (18) with updated path set $\hat{\mathcal{R}}^\pi$ gives q^* in (39). Algorithm 3 with arc travel times $t_a(x_a(q^*))$ gives path set $\mathcal{R}^{\pi+} = \{r_1, r_2, r_6, r_9, r_{10}\} \subset \hat{\mathcal{R}}^\pi$. If Algorithm 3 selects r_3 instead of r_2 at $w = 0.01$, or skips path r_6 at $w = 0.07$, it still holds that $\mathcal{R}^{\pi+} \subset \hat{\mathcal{R}}^\pi$. Algorithm BTA terminates, and Lemma 10 verifies optimality for the full path set. Algorithm 1 then gives w^* in (32). □



(a) Parametric shortest path solution with zero initial flow.



(b) Parametric shortest path solution after the first flow update.



(c) Parametric shortest path solution for flow $x(q^*)$.

Figure 5 The three parametric shortest path solutions produced by Algorithm 3 in Example 6. Thin gray lines are the bicriteria costs of all ten paths and thick colored pieces their lower envelope. Panel (a) shows the first path set $\hat{\mathcal{R}}^\pi$. Panel (b) shows the parametric shortest path solution for the updated flow $x(\hat{q})$; path r_6 is optimal only at the marked singleton $w = 0.07$. Panel (c) shows the final parametric shortest path solution for the final flow $x(q^*)$, when every parametric shortest path is already in $\hat{\mathcal{R}}^\pi$.

5. Numerical Results

In this section we compare Algorithm BTA with the methods of Dafermos (1972), Daganzo (1983), Dial (1996), Marcotte et al. (1996), and Xie et al. (2026). Specifically, from Dafermos (1972) we use the path-correction algorithm for the multiclass problem, in its TNTP surrogate cost mode. From Daganzo (1983) we use the stochastic approximation algorithm, with the feasible Cesàro average. From Dial (1996) we use T2-ETA, which calls T2-MPA as its minimum-path assignment subproblem. From Marcotte et al. (1996) we use the parametric shortest path descent with the paper's exact line search (not its Armijo variants). From Xie et al. (2026) we use SBA rather than the Frank-Wolfe or T2-θ algorithms that accompany it.

Table EC.1 gives summary network data such as the numbers of nodes, arcs, zones, OD pairs, and total demand, for each benchmark network used in the numerical tests. Link travel times are

given by the BPR function $t_a(x_a) = t_a^0\big(1 + \alpha_a(x_a/c_a)^{\beta_a}\big)$, with α_a and β_a given in Table EC.2. The arc flow monetary costs are given by $s_a = \alpha_0 x_a^{\mathrm{SO}} t_a'(x_a^{\mathrm{SO}})$, called *marginal-cost tolls*, rounded to the nearest 0.01 for every $a \in \mathcal{A}$, with $\alpha_0 = 1$ (except for the results discussed in Section 5.3) and x_a^{SO} being the system optimal arc flows for the same instance. Tests were performed with two types of instances: smooth instances, meaning that Υ is smooth (a sufficient condition is that G^π has a positive density on $[\underline{w}^\pi, \overline{w}^\pi]$, as the methods of Marcotte et al. 1996 and Xie et al. 2026 require), and nonsmooth instances. For the smooth instances, all methods used $w \sim \mathcal{U}[6, 30]$, and for the nonsmooth instances w followed a "staircase" distribution with equal point masses at $w = 0, 2.5, 5, 7.5, 10$.

All methods, except the SBA method, are implemented in C11 with POSIX threads, and compiled via Clang. The SBA code, obtained from the authors of Xie et al. (2026), was written in C++14, built from GCC 12.3.0. The reported computation times exclude the computation of x^{SO} to determine input arc flow monetary costs s_a. We ran BTA with two solvers for solving problem (18) with subsets of paths $\hat{\mathcal{R}}^\pi$ in Line 10 of Algorithm 2: The FISTA method of Sujanani and Monteiro (2024) was used for the smooth instances, and the two-cut adaptive proximal bundle method (Ad-GPB) of Monteiro and Zhang (2024) was used for the nonsmooth instances. Table EC.2 gives the parameters used for these solvers.

5.1. Benchmarking Metrics

For each instance i and method m, let $q_{i,m}(t')$ denote the incumbent solution at CPU time t', and let

$$V_{i,m}(t) \;:=\; \min_{t' \le t} \; \Upsilon_i\big(q_{i,m}(t')\big) \tag{40}$$

denote the best objective value attained by time t. For each instance i we also compute a lower bound on the optimal objective value Υ_i^*, as follows. It follows from convexity of Υ_i that

$$\Upsilon_i^* \;\ge\; \Upsilon_i(q) + \min\left\{\langle g, y - q\rangle \;:\; y \in \prod_{\pi \in \Pi} \Delta_{\mathcal{R}^\pi}\right\} \tag{41}$$

for any feasible q and any subgradient $g \in \partial\Upsilon_i(q)$ at q. For instance i, let L_i denote the largest lower bound of (41) over all iterates q produced by all methods m. Thus, L_i is the best lower bound on Υ_i^* computed during the test. Let q_i^0 denote an initial feasible solution for instance i obtained after Line 4 of Algorithm 2. Then the relative gap for any method m on instance i at time t is

$$\gamma_{i,m}(t) \;:=\; \frac{V_{i,m}(t) - L_i}{\Upsilon_i(q_i^0) - L_i}. \tag{42}$$

Thus, $\gamma_{i,\mathrm{BTA}}(0) = 1$. Note that the denominator does not depend on the method m, and is used only for scaling. For any tolerance $\varepsilon_i > 0$ and maximum compute time T_i for instance i, let

$$t_{i,m} := \begin{cases} \inf\{t \in [0,T_i] : \gamma_{i,m}(t) \le \varepsilon_i\} & \text{if } \{t \in [0,T_i] : \gamma_{i,m}(t) \le \varepsilon_i\} \neq \emptyset \\ T_i & \text{otherwise} \end{cases}$$

denote the time at which method m first reaches the tolerance for instance i, capped at T_i. We used $\varepsilon_i = 10^{-5}$ for smooth instances and $\varepsilon_i = 10^{-3}$ for nonsmooth instances. We used $T_i = 3600$s for all instances, except for the smooth Goldcoast, Chicago Regional, and Philadelphia instances, for which $T_i = 21600$s, and the nonsmooth Goldcoast, Chicago Regional, and Philadelphia instances (each with more than a million OD records), for which $T_i = 86400$s. As a time-dependent aggregate performance metric over set $\mathcal{I}$ of instances, we compute the fraction of instances for which method m finds a sufficiently good solution by time t, given by

$$P_m(t) := \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} \mathbb{1}\left\{\gamma_{i,m}(\min\{t, T_i\}) \le \varepsilon_i\right\}.$$

Other aggregate performance metrics for each method m are

$$g_m = \left[\prod_{i \in \mathcal{I}} (t_{i,m} + 1)\right]^{1/|\mathcal{I}|} - 1, \qquad r_m = \frac{g_m}{\min_{m'} g_{m'}},$$
$$h_m = \frac{1}{|\mathcal{I}|} \sum_{i \in \mathcal{I}} \mathbb{1}\left\{\gamma_{i,m}(T_i) \le \varepsilon_i,\ t_{i,m} = \min_{m'} t_{i,m'}\right\},$$

the shifted geometric mean capped completion time, its normalized value, and the fraction of all instances for which method m reaches the tolerance ε_i by time T_i and is the fastest.

5.2. Convergence Performance on Benchmark Networks

Table 1 and Figure 6 compare BTA with the methods of Daganzo (1983), Dial (1996), Marcotte et al. (1996) and Xie et al. (2026) for smooth instances. No method reached the target tolerance ε_i by time T_i for the Goldcoast, Chicago Regional, or Philadelphia instances, and BTA was the fastest on all 14 remaining instances.

Table 2 compares BTA with the method of Dafermos (1972) for nonsmooth instances for which w has finite support. Dafermos reached the target tolerance ε_i by time T_i for all 17 instances, while BTA reached ε_i by time T_i for 15 of the 17 instances (excluding Goldcoast and Philadelphia). Tables EC.3 and EC.4 in Appendix EC.4 report the per-instance computation times for both comparisons.

Table 3 reports the solution times $t_{i,\mathrm{BTA}}$ and final gaps $\gamma_{i,\mathrm{BTA}}(T_i)$ for BTA. If $t_{i,\mathrm{BTA}} < T_i$ then $\gamma_{i,\mathrm{BTA}}(T_i) \le \varepsilon_i$, and if $t_{i,\mathrm{BTA}} = T_i$ then $\gamma_{i,\mathrm{BTA}}(T_i) > \varepsilon_i$.

Table 1 Comparison of BTA and the methods of Daganzo (1983), Dial (1996), Marcotte et al. (1996), and Xie et al. (2026) for smooth instances, in terms of the shifted geometric mean completion time g_m, its normalized value r_m, the fastest share h_m, and the number of instances i solved in time T_i.

Method	g_m	r_m	h_m	Instances i solved in time T_i
Daganzo	3120.898	250.622	0.000	1/17
Dial	196.485	15.779	0.000	12/17
Marcotte–Zhu	68.737	5.520	0.000	13/17
SBA	59.983	4.817	0.000	13/17
BTA	12.453	1.000	0.824	14/17

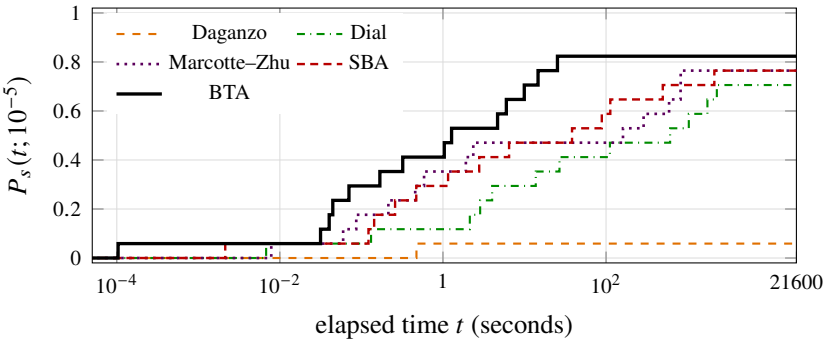


Figure 6 Plot of $P_m(t)$ for BTA and the methods of Daganzo (1983), Dial (1996), Marcotte et al. (1996), and Xie et al. (2026) over all 17 smooth instances. BTA reaches 0.82, missing only Goldcoast, Chicago Regional, and Philadelphia, for which no method reaches the target tolerance; SBA reaches 0.76, Marcotte-Zhu 0.76, Dial 0.71, and Daganzo 0.06.

Table 2 Comparison of BTA and the method of Dafermos (1972) for instances with finite support for w, in terms of the shifted geometric mean completion time g_m, its normalized value r_m, the fastest share h_m, and the number of instances i solved in time T_i.

Method	g_m	r_m	h_m	Instances i solved in time T_i
Dafermos	25.641	1.000	0.882	17/17
BTA	204.737	7.985	0.118	15/17

5.3. Impact of Tolls on Path Diversity

It is of interest to determine how much the introduction of a second criterion such as tolls affect the diversity of paths used in equilibrium. We compare the fraction of OD pairs with positive flow in equilibrium on more than one path in four settings: no tolls, lower marginal-cost tolls with $\alpha_0 = 1$, higher marginal-cost tolls with $\alpha_0 = 18$, and selective capacity-UE tolls (specified in Appendix EC.4). The no-toll case is the single-criterion baseline. Table 4 reports the fractions for four networks. The fraction of OD pairs with positive flow on more than one path is greater with a second criterion (tolls) than with a single criterion (time), and in most, but not all, cases, the fraction tends to increase as the tolls increase.

Table 3 Solution times $t_{i,\mathrm{BTA}}$ and final gaps $\gamma_{i,\mathrm{BTA}}(T_i)$ for BTA.

	Smooth, $w \sim \mathcal{U}[6, 30]$		Nonsmooth, $w \sim$ staircase	
Network	Time	Gap	Time	Gap
Braess	< 0.001	$\leq 10^{-5}$	0.016	$\leq 10^{-3}$
Sioux Falls	0.032	$\leq 10^{-5}$	0.191	$\leq 10^{-3}$
Berlin Tiergarten	0.040	$\leq 10^{-5}$	7.915	$\leq 10^{-3}$
Friedrichshain	0.045	$\leq 10^{-5}$	15.671	$\leq 10^{-3}$
Anaheim	0.321	$\leq 10^{-5}$	140.964	$\leq 10^{-3}$
Berlin Prenzlauerberg	0.168	$\leq 10^{-5}$	39.797	$\leq 10^{-3}$
Berlin Mitte	0.070	$\leq 10^{-5}$	33.597	$\leq 10^{-3}$
Winnipeg-Asym	1.275	$\leq 10^{-5}$	42.728	$\leq 10^{-3}$
Berlin M.–P.–F.	1.038	$\leq 10^{-5}$	275.643	$\leq 10^{-3}$
Terrassa-Asym	9.935	$\leq 10^{-5}$	280.418	$\leq 10^{-3}$
Hessen-Asym	14.657	$\leq 10^{-5}$	314.519	$\leq 10^{-3}$
Winnipeg	4.673	$\leq 10^{-5}$	84.343	$\leq 10^{-3}$
Barcelona	5.987	$\leq 10^{-5}$	2093.585	$\leq 10^{-3}$
Berlin Center	25.352	$\leq 10^{-5}$	2014.743	$\leq 10^{-3}$
Goldcoast	21600	1.09×10^{-2}	43200	5.32×10^{-3}
Philadelphia	21600	5.06×10^{-4}	43200	1.65×10^{-2}
Chicago Regional	21600	2.03×10^{-1}	20352.302	$\leq 10^{-3}$

Table 4 The fraction of OD pairs with positive flow in equilibrium on more than one path under four tolling schemes. Table EC.6 in Appendix EC.4 also gives elapsed times, final gaps, and used-path counts for the same instances.

Network	Instance	No toll	MC $\alpha_0 = 1$	MC $\alpha_0 = 18$	Capacity-UE
Sioux Falls	Smooth	0.265	0.394	0.286	0.345
	Nonsmooth	0.222	0.570	0.697	0.540
Anaheim	Smooth	0.286	0.583	0.762	0.323
	Nonsmooth	0.337	0.764	0.925	0.698
Berlin Tiergarten	Smooth	0.222	0.484	0.630	0.247
	Nonsmooth	0.238	0.618	0.835	0.315
Winnipeg-Asym	Smooth	0.788	0.827	0.722	0.789
	Nonsmooth	0.786	0.882	0.919	0.829

6. Conclusion

The method for the bicriteria traffic assignment problem proposed in this paper has a number of advantages. First, it can be used with any distribution for the weight that trades off the two criteria, whereas previously proposed methods were either restricted to a distribution with finite support or a distribution with a positive density on an interval. Second, with the proposed method an equilibrium can be computed by solving a finite dimensional convex optimization problem. This advantage contributes to the proposed algorithm having better empirical performance than previously proposed algorithms, with the exception that the algorithm of Dafermos (1972), designed for instances with a finite number of user classes (the distribution has finite support), often performs better on such instances. A closer look at the relative performance of the algorithms on these instances reveals a shortcoming of the proposed algorithm — it spends too much time solving the

restricted problem with a small subset of paths to the accuracy required of the final solution, instead of stopping the iteration earlier to generate new paths. Such an enhancement requires a different algorithm, and is the topic of further research.

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Electronic Companion

EC.1. Proofs for Results in Section 4.1

Lemma 1 *For any $\pi \in \Pi$, consider two total orders $\prec$ and $\lessdot$ on $\mathcal{R}^\pi$ such that for any $r, r' \in \mathcal{R}^\pi$ it holds that if $r \prec r'$ or $r \lessdot r'$ then $S^\pi_r \geq S^\pi_{r'}$. Consider any $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$ and any $q^\pi \in \Delta_{\hat{\mathcal{R}}^\pi}$. For each $r \in \hat{\mathcal{R}}^\pi$, let*

$$p^\pi_r \;:=\; \sum_{\{r' \in \hat{\mathcal{R}}^\pi \,:\, r' \preceq r\}} q^\pi_{r'} \qquad \text{and} \qquad \dot{p}^\pi_r \;:=\; \sum_{\{r' \in \hat{\mathcal{R}}^\pi \,:\, r' \lessdot r\}} q^\pi_{r'}$$

Then

$$\sum_{r \in \hat{\mathcal{R}}^\pi} S^\pi_r \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{p^\pi_{r-1}, p^\pi_r}(y) \;=\; \sum_{r \in \hat{\mathcal{R}}^\pi} S^\pi_r \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{\dot{p}^\pi_{r-1}, \dot{p}^\pi_r}(y) \tag{EC.1}$$

and thus $\phi(q)$ is invariant with respect to the chosen order.

Proof. Consider any $r \in \hat{\mathcal{R}}^\pi$. Let $r' \in \hat{\mathcal{R}}^\pi$ satisfy $S^\pi_{r'} = S^\pi_r$ and $S^\pi_{r'''} > S^\pi_r$ for all $r''' \in \hat{\mathcal{R}}^\pi$ such that $r''' \prec r'$, and let $r'' \in \hat{\mathcal{R}}^\pi$ satisfy $S^\pi_{r''} = S^\pi_r$ and $S^\pi_{r'''} < S^\pi_r$ for all $r''' \in \hat{\mathcal{R}}^\pi$ such that $r'' \prec r'''$. Similarly, let $\dot{r}' \in \hat{\mathcal{R}}^\pi$ satisfy $S^\pi_{\dot{r}'} = S^\pi_r$ and $S^\pi_{r'''} > S^\pi_r$ for all $r''' \in \hat{\mathcal{R}}^\pi$ such that $r''' \lessdot \dot{r}'$, and let $\dot{r}'' \in \hat{\mathcal{R}}^\pi$ satisfy $S^\pi_{\dot{r}''} = S^\pi_r$ and $S^\pi_{r'''} < S^\pi_r$ for all $r''' \in \hat{\mathcal{R}}^\pi$ such that $\dot{r}'' \lessdot r'''$. Then $\{r''' \in \hat{\mathcal{R}}^\pi \,:\, S^\pi_{r'''} > S^\pi_r\} = \{r''' \in \hat{\mathcal{R}}^\pi \,:\, r''' \prec r'\} = \{r''' \in \hat{\mathcal{R}}^\pi \,:\, r''' \lessdot \dot{r}'\}$, $\{r''' \in \hat{\mathcal{R}}^\pi \,:\, S^\pi_{r'''} = S^\pi_r\} = \{r''' \in \hat{\mathcal{R}}^\pi \,:\, r' \preceq r''' \preceq r''\} = \{r''' \in \hat{\mathcal{R}}^\pi \,:\, \dot{r}' \lessdot r''' \lessdot \dot{r}''\}$, and $\{r''' \in \hat{\mathcal{R}}^\pi \,:\, S^\pi_{r'''} < S^\pi_r\} = \{r''' \in \hat{\mathcal{R}}^\pi \,:\, r'' \prec r'''\} = \{r''' \in \hat{\mathcal{R}}^\pi \,:\, \dot{r}'' \lessdot r'''\}$. Thus,

$$p^\pi_{r'-1} \;=\; \sum_{\{r''' \in \hat{\mathcal{R}}^\pi \,:\, r''' \prec r'\}} q^\pi_{r'''} \;=\; \sum_{\{r''' \in \hat{\mathcal{R}}^\pi \,:\, r''' \lessdot \dot{r}'\}} q^\pi_{r'''} \;=\; \dot{p}^\pi_{\dot{r}'-1}$$

and

$$p^\pi_{r''} \;=\; p^\pi_{r'-1} + \sum_{\{r''' \in \hat{\mathcal{R}}^\pi \,:\, r' \preceq r''' \preceq r''\}} q^\pi_{r'''} \;=\; \dot{p}^\pi_{\dot{r}'-1} + \sum_{\{r''' \in \hat{\mathcal{R}}^\pi \,:\, \dot{r}' \lessdot r''' \lessdot \dot{r}''\}} q^\pi_{r'''} \;=\; \dot{p}^\pi_{\dot{r}''}$$

Hence,

$$\begin{aligned}
\sum_{\{r''' \in \hat{\mathcal{R}}^\pi \,:\, S^\pi_{r'''} = S^\pi_r\}} S^\pi_{r'''} \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{p^\pi_{r'''-1}, p^\pi_{r'''}}(y) \;&=\; S^\pi_{r'} \sum_{\{r''' \in \hat{\mathcal{R}}^\pi \,:\, r' \preceq r''' \preceq r''\}} \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{p^\pi_{r'''-1}, p^\pi_{r'''}}(y) \\
&\overset{(10)}{=}\; S^\pi_r \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{p^\pi_{r'-1}, p^\pi_{r''}}(y) \\
&=\; S^\pi_r \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{\dot{p}^\pi_{\dot{r}'-1}, \dot{p}^\pi_{\dot{r}''}}(y) \\
&\overset{(10)}{=}\; S^\pi_r \sum_{\{r''' \in \hat{\mathcal{R}}^\pi \,:\, \dot{r}' \lessdot r''' \lessdot \dot{r}''\}} \int_{\mathbb{R}} y \; \mathrm{d}G^\pi_{\dot{p}^\pi_{r'''-1}, \dot{p}^\pi_{r'''}}(y)
\end{aligned}$$

$$= \sum_{\{r''' \in \hat{\mathcal{R}}^\pi : S^\pi_{r'''} = S^\pi_r\}} S^\pi_{r'''} \int_{\mathbb{R}} y \, \mathrm{d}G^\pi_{\dot{p}^\pi_{r'''_{-1}}, \dot{p}^\pi_{r'''}}(y)$$

The equality holds for every value S^π_r and every $\pi \in \Pi$, and therefore (EC.1) holds. □

Lemma 2 *The function $\varphi : [0,1] \to \mathbb{R}$ is convex and Lipschitz continuous, and $\partial\varphi(p) = G^{-1}(p)$ for every $p \in [0,1]$.*

Proof. Recall that $G^{-1}(p) \neq \emptyset$ for all $p \in [0,1]$. We first show that for every $\hat{p} \in [0,1]$ and every $\hat{w} \in G^{-1}(\hat{p})$, $\hat{w}$ is a subgradient of φ at $\hat{p}$, which establishes that φ is convex and that $G^{-1}(\hat{p}) \subset \partial\varphi(\hat{p})$. For every $p \in [\hat{p}, 1]$ it holds that

$$\varphi(p) - \varphi(\hat{p}) \;=\; \int_{\mathbb{R}} w \, \mathrm{d}G_{0,p}(w) - \int_{\mathbb{R}} w \, \mathrm{d}G_{0,\hat{p}}(w) \;=\; \int_{\mathbb{R}} w \, \mathrm{d}G_{\hat{p},p}(w) \;\geq\; \int_{\mathbb{R}} \hat{w} \, \mathrm{d}G_{\hat{p},p}(w) \;=\; \hat{w}(p - \hat{p}).$$

Also, for every $p \in [0, \hat{p}]$ it holds that

$$\varphi(p) - \varphi(\hat{p}) \;=\; \int_{\mathbb{R}} w \, \mathrm{d}G_{0,p}(w) - \int_{\mathbb{R}} w \, \mathrm{d}G_{0,\hat{p}}(w) \;=\; -\int_{\mathbb{R}} w \, \mathrm{d}G_{p,\hat{p}}(w) \;\geq\; -\int_{\mathbb{R}} \hat{w} \, \mathrm{d}G_{p,\hat{p}}(w) \;=\; \hat{w}(p - \hat{p}).$$

Thus $\varphi(p) - \varphi(\hat{p}) \geq \hat{w}(p - \hat{p})$ for all $p \in [0,1]$. It follows that φ is convex and that $G^{-1}(\hat{p}) \subset \partial\varphi(\hat{p})$ for all $\hat{p} \in [0,1]$. Also, for any $0 \leq \hat{p} \leq p \leq 1$ it holds that

$$\varphi(p) - \varphi(\hat{p}) \;=\; \int_{\underline{w}}^{\overline{w}} w \, \mathrm{d}G_{\hat{p},p}(w) \;\leq\; \overline{w}(p - \hat{p})$$

and thus φ is Lipschitz continuous with Lipschitz constant $\overline{w}$.

Next we show that $\partial\varphi(\hat{p}) \subset G^{-1}(\hat{p})$ for all $\hat{p} \in [0,1]$. Consider any $\hat{w} \notin G^{-1}(\hat{p})$. Then, either $\hat{w} < \min\{w \in G^{-1}(\hat{p})\}$ or $\hat{w} > \max\{w \in G^{-1}(\hat{p})\}$.

First, consider the case in which $\hat{w} < \min\{w \in G^{-1}(\hat{p})\}$, that is, there is a $\check{w} > \hat{w}$ such that $\check{p} := G(\check{w}) < \hat{p}$. Then

$$\varphi(\check{p}) - \varphi(\hat{p}) \;=\; -\int_{\mathbb{R}} w \, \mathrm{d}G_{\check{p},\hat{p}}(w) \;\leq\; -\int_{\mathbb{R}} \check{w} \, \mathrm{d}G_{\check{p},\hat{p}}(w) \;=\; -\check{w}(\hat{p} - \check{p}) \;<\; \hat{w}(\check{p} - \hat{p})$$

and thus $\hat{w} \notin \partial\varphi(\hat{p})$. Next, consider the case in which $\hat{w} > \max\{w \in G^{-1}(\hat{p})\}$, that is, there is a $\bar{w} < \hat{w}$ such that $\bar{p} := G(\bar{w}) > \hat{p}$. Then

$$\varphi(\bar{p}) - \varphi(\hat{p}) \;=\; \int_{\mathbb{R}} w \, \mathrm{d}G_{\hat{p},\bar{p}}(w) \;\leq\; \int_{\mathbb{R}} \bar{w} \, \mathrm{d}G_{\hat{p},\bar{p}}(w) \;=\; \bar{w}(\bar{p} - \hat{p}) \;<\; \hat{w}(\bar{p} - \hat{p})$$

and thus $\hat{w} \notin \partial\varphi(\hat{p})$. Hence $\partial\varphi(\hat{p}) \subset G^{-1}(\hat{p})$ for all $\hat{p} \in [0,1]$.

Therefore it has been established that $\partial\varphi(p) = G^{-1}(p)$ for every $p \in [0,1]$. □

Lemma 3 *The function $\phi : \prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi} \to \mathbb{R}$ is convex and continuous, and*

$$\partial\phi(q) \;=\; \prod_{\pi\in\Pi}\left(D^\pi \sum_{\{r\in\hat{\mathcal{R}}^\pi : r+1\in\hat{\mathcal{R}}^\pi\}} \left(S^\pi_r - S^\pi_{r+1}\right)\left(\mathbb{1}\left\{r' \preceq r\right\},\, r'\in\hat{\mathcal{R}}^\pi\right)\left(G^\pi\right)^{-1}\left(L^\pi_r(q^\pi)\right) + \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)\right)$$

Proof. It follows from $S^\pi_r \geq S^\pi_{r+1}$ and Lemma 2 that ϕ is convex and continuous. For each $\pi\in\Pi$, let $\phi^\pi : \Delta_{\hat{\mathcal{R}}^\pi} \to \mathbb{R}$ be given by

$$\phi^\pi(q^\pi) \;:=\; D^\pi \sum_{r\in\hat{\mathcal{R}}^\pi}\left[\mathbb{1}\{r+1\in\hat{\mathcal{R}}^\pi\}\left(S^\pi_r - S^\pi_{r+1}\right)\phi^\pi_r(q^\pi) + \mathbb{1}\{r+1\notin\hat{\mathcal{R}}^\pi\}S^\pi_{\min}\varphi^\pi(1)\right]$$

For each $\pi\in\Pi$, it holds that

$$\bigcap_{r\in\hat{\mathcal{R}}^\pi} \mathrm{ri}\big(\mathrm{dom}(\phi^\pi_r)\big) \;=\; \Delta^\circ_{\hat{\mathcal{R}}^\pi} \;\neq\; \varnothing$$

and thus it follows from the Moreau-Rockafellar Theorem (Rockafellar 1997, Theorem 23.8) that

$$\begin{aligned}\partial\phi^\pi(q^\pi) \;&=\; D^\pi \sum_{\{r\in\hat{\mathcal{R}}^\pi : r+1\in\hat{\mathcal{R}}^\pi\}} \left(S^\pi_r - S^\pi_{r+1}\right)\partial\phi^\pi_r(q^\pi) + \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)\\ &=\; D^\pi \sum_{\{r\in\hat{\mathcal{R}}^\pi : r+1\in\hat{\mathcal{R}}^\pi\}} \left(S^\pi_r - S^\pi_{r+1}\right)\left(\mathbb{1}\left\{r' \preceq r\right\},\, r'\in\hat{\mathcal{R}}^\pi\right)\left(G^\pi\right)^{-1}\left(L^\pi_r(q^\pi)\right) + \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)\end{aligned}$$

Then it follows from $\phi(q) = \sum_{\pi\in\Pi}\phi^\pi(q^\pi)$ being separable that

$$\begin{aligned}\partial\phi(q) \;&=\; \prod_{\pi\in\Pi}\partial\phi^\pi(q^\pi)\\ &=\; \prod_{\pi\in\Pi}\left(D^\pi \sum_{\{r\in\hat{\mathcal{R}}^\pi : r+1\in\hat{\mathcal{R}}^\pi\}} \left(S^\pi_r - S^\pi_{r+1}\right)\left(\mathbb{1}\left\{r' \preceq r\right\},\, r'\in\hat{\mathcal{R}}^\pi\right)\left(G^\pi\right)^{-1}\left(L^\pi_r(q^\pi)\right) + \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)\right)\end{aligned}$$

□

Lemma 4 *The function ψ is convex and continuous on $\prod_{\pi\in\Pi}\Delta_{\hat{\mathcal{R}}^\pi}$, and*

$$\partial\psi(q) \;=\; \left\{ g = \left(g^\pi_r, \pi\in\Pi, r\in\hat{\mathcal{R}}^\pi\right) \,:\, g^\pi_r = D^\pi \sum_{a\in\mathcal{A}^\pi_r} \tau_a,\; \tau_a \in \left[t^-_a(x_a(q)), t^+_a(x_a(q))\right]\;\forall\, a\in\mathcal{A}\right\} + \prod_{\pi\in\Pi}\mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$$

with $\left[t^-_a(0), t^+_a(0)\right] := \left(-\infty, t^+_a(0)\right]$.

Proof. It is convenient to extend the domain of ψ before establishing its subdifferential: Let

$$Q \;:=\; \left\{ q = \left(q^\pi_r, \pi\in\Pi, r\in\hat{\mathcal{R}}^\pi\right) \,:\, x_a(q)\geq 0\;\forall\, a\in\mathcal{A}\right\}$$

Note that $\prod_{\pi\in\Pi}\Delta_{\hat{\mathcal{R}}^\pi} \subset \prod_{\pi\in\Pi}\mathbb{R}_+^{\hat{\mathcal{R}}^\pi} \subset Q \subset \prod_{\pi\in\Pi}\mathbb{R}^{\hat{\mathcal{R}}^\pi}$. Let $\psi : \prod_{\pi\in\Pi}\mathbb{R}^{\hat{\mathcal{R}}^\pi} \mapsto \mathbb{R}\cup\{\infty\}$ be given by (17) if $q \in \prod_{\pi\in\Pi}\Delta_{\hat{\mathcal{R}}^\pi}$, and $\psi(q) = \infty$ if $q \notin \prod_{\pi\in\Pi}\Delta_{\hat{\mathcal{R}}^\pi}$, that is, $\psi(q) = \hat{\psi}(q) + \sum_{\pi\in\Pi}\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$ for all $q \in \prod_{\pi\in\Pi}\mathbb{R}^{\hat{\mathcal{R}}^\pi}$, where

$$\hat{\psi}(q) \;:=\; \begin{cases} \sum_{a\in\mathcal{A}}\int_0^{x_a(q)} t_a(y)\,\mathrm{d}y & \text{if } q \in Q \\ \infty & \text{if } q \notin Q \end{cases}$$

and, for each $\pi \in \Pi$, $\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}} : \mathbb{R}^{\hat{\mathcal{R}}^\pi} \mapsto \mathbb{R}\cup\{\infty\}$ is the indicator function of $\Delta_{\hat{\mathcal{R}}^\pi}$ given by

$$\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi) \;:=\; \begin{cases} 0 & \text{if } q^\pi \in \Delta_{\hat{\mathcal{R}}^\pi} \\ \infty & \text{if } q^\pi \notin \Delta_{\hat{\mathcal{R}}^\pi} \end{cases}$$

Note that

$$\mathrm{ri}\left(\mathrm{dom}\left(\hat{\psi}\right)\right) \cap \mathrm{ri}\left(\mathrm{dom}\left(\sum_{\pi\in\Pi}\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}\right)\right) \;=\; \prod_{\pi\in\Pi}\Delta^\circ_{\hat{\mathcal{R}}^\pi} \;\neq\; \emptyset$$

and thus it follows from the Moreau-Rockafellar Theorem (Rockafellar 1997, Theorem 23.8) and $\psi(q) = \hat{\psi}(q) + \sum_{\pi\in\Pi}\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$ that $\partial\psi(q) = \partial\hat{\psi}(q) + \partial\sum_{\pi\in\Pi}\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$. Also, it follows from $\sum_{\pi\in\Pi}\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$ being separable that $\partial\sum_{\pi\in\Pi}\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi) = \prod_{\pi\in\Pi}\partial\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi) = \prod_{\pi\in\Pi}\mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$.

Next we determine $\partial\hat{\psi}(q)$. Consider any $\hat{q} \in Q$ and any $\hat{g}$ such that $\hat{g}^\pi_r = D^\pi\sum_{a\in\mathcal{A}^\pi_r}\tau_a$, $\tau_a \in \left[t_a^-(x_a(\hat{q})), t_a^+(x_a(\hat{q}))\right]\ \forall\, a \in \mathcal{A}$. For any $q \in Q$ and $a \in \mathcal{A}$, first consider the case in which $x_a(q) \geq x_a(\hat{q})$. Then

$$\begin{aligned}\int_0^{x_a(q)} t_a(y)\,\mathrm{d}y - \int_0^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y \;&=\; \int_{x_a(\hat{q})}^{x_a(q)} t_a(y)\,\mathrm{d}y \\ &\geq\; t_a^+(x_a(\hat{q}))\,[x_a(q) - x_a(\hat{q})] \;\geq\; \tau_a\,[x_a(q) - x_a(\hat{q})]\end{aligned}$$

Next consider the case in which $x_a(q) < x_a(\hat{q})$. Then

$$\begin{aligned}\int_0^{x_a(q)} t_a(y)\,\mathrm{d}y - \int_0^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y \;&=\; -\int_{x_a(q)}^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y \\ &\geq\; -t_a^-(x_a(\hat{q}))\,[x_a(\hat{q}) - x_a(q)] \;\geq\; \tau_a\,[x_a(q) - x_a(\hat{q})]\end{aligned}$$

Thus,

$$\begin{aligned}\hat{\psi}(q) - \hat{\psi}(\hat{q}) \;&=\; \sum_{a\in\mathcal{A}}\int_0^{x_a(q)} t_a(y)\,\mathrm{d}y - \sum_{a\in\mathcal{A}}\int_0^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y \\ &\geq\; \sum_{a\in\mathcal{A}}\tau_a\,[x_a(q) - x_a(\hat{q})]\end{aligned}$$

$$= \sum_{a\in\mathcal{A}} \tau_a \left[\sum_{\pi\in\Pi} D^\pi \sum_{r\in\hat{\mathcal{R}}_a^\pi} q_r^\pi - \sum_{\pi\in\Pi} D^\pi \sum_{r\in\hat{\mathcal{R}}_a^\pi} \hat{q}_r^\pi \right]$$
$$= \sum_{\pi\in\Pi} \sum_{r\in\hat{\mathcal{R}}^\pi} D^\pi \sum_{a\in\mathcal{A}_r^\pi} \tau_a \left(q_r^\pi - \hat{q}_r^\pi\right) = \hat{g}^\top (q - \hat{q})$$

and hence $\hat{g} \in \partial\hat{\psi}(\hat{q})$. Therefore,

$$\partial\hat{\psi}(q) \supset \left\{ g = \left(g_r^\pi, \pi\in\Pi, r\in\hat{\mathcal{R}}^\pi\right) : g_r^\pi = D^\pi \sum_{a\in\mathcal{A}_r^\pi} \tau_a,\ \tau_a \in \left[t_a^-(x_a(q)), t_a^+(x_a(q))\right]\ \forall\, a\in\mathcal{A} \right\}$$

Next, consider any $\hat{g} \notin \left\{ g = \left(g_r^\pi, \pi\in\Pi, r\in\hat{\mathcal{R}}^\pi\right) : g_r^\pi = D^\pi \sum_{a\in\mathcal{A}_r^\pi} \tau_a,\ \tau_a \in \left[t_a^-(x_a(\hat{q})), t_a^+(x_a(\hat{q}))\right]\ \forall\, a\in\mathcal{A} \right\}$. Let $\mathcal{A}_0 := \{a\in\mathcal{A} : x_a(\hat{q}) = 0\}$, $\mathcal{A}_= := \{a\in\mathcal{A} : x_a(\hat{q}) > 0,\ t_a^-(x_a(\hat{q})) = t_a^+(x_a(\hat{q}))\}$, and $\mathcal{A}_+ := \{a\in\mathcal{A} : x_a(\hat{q}) > 0,\ t_a^-(x_a(\hat{q})) < t_a^+(x_a(\hat{q}))\}$. Then

$$\hat{z} := \min_{\theta,\tau} \sum_{\pi\in\Pi} \sum_{r\in\hat{\mathcal{R}}^\pi} \theta_r^\pi \tag{EC.2}$$
$$\text{s.t. } \theta_r^\pi \geq \hat{g}_r^\pi - D^\pi \sum_{a\in\mathcal{A}_r^\pi} \tau_a \qquad \forall\,\pi\in\Pi,\ \forall\, r\in\hat{\mathcal{R}}^\pi \tag{EC.3}$$
$$\theta_r^\pi \geq D^\pi \sum_{a\in\mathcal{A}_r^\pi} \tau_a - \hat{g}_r^\pi \qquad \forall\,\pi\in\Pi,\ \forall\, r\in\hat{\mathcal{R}}^\pi \tag{EC.4}$$
$$\tau_a \geq t_a^-(x_a(\hat{q})) \qquad \forall\, a\in\mathcal{A}_+ \tag{EC.5}$$
$$\tau_a \leq t_a^+(x_a(\hat{q})) \qquad \forall\, a\in\mathcal{A}_0\cup\mathcal{A}_+ \tag{EC.6}$$
$$\tau_a = t_a(x_a(\hat{q})) \qquad \forall\, a\in\mathcal{A}_= \tag{EC.7}$$

satisfies $\hat{z} > 0$. Let $\eta_r^\pi \geq 0$ denote the dual variables associated with (EC.3), let $\zeta_r^\pi \geq 0$ denote the dual variables associated with (EC.4), let $\lambda_a^- \geq 0$ denote the dual variables associated with (EC.5), let $\lambda_a^+ \geq 0$ denote the dual variables associated with (EC.6), and let $\lambda_a^=$ denote the dual variables associated with (EC.7). Then

$$\hat{z} = \max_{\eta,\zeta,\lambda^-,\lambda^+,\lambda^=} \left\{ \sum_{\pi\in\Pi} \sum_{r\in\hat{\mathcal{R}}^\pi} \hat{g}_r^\pi \left(\eta_r^\pi - \zeta_r^\pi\right) + \sum_{a\in\mathcal{A}_+} t_a^-(x_a(\hat{q}))\lambda_a^- - \sum_{a\in\mathcal{A}_0\cup\mathcal{A}_+} t_a^+(x_a(\hat{q}))\lambda_a^+ + \sum_{a\in\mathcal{A}_=} t_a(x_a(\hat{q}))\lambda_a^= \right\} \tag{EC.8}$$
$$\text{s.t. } \eta_r^\pi + \zeta_r^\pi = 1 \qquad \forall\,\pi\in\Pi,\ \forall\, r\in\hat{\mathcal{R}}^\pi \tag{EC.9}$$
$$-\lambda_a^+ + \sum_{\pi\in\Pi} \sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi \left(\eta_r^\pi - \zeta_r^\pi\right) = 0 \qquad \forall\, a\in\mathcal{A}_0 \tag{EC.10}$$

$$\left(\lambda_a^- - \lambda_a^+\right) + \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\eta_r^\pi - \zeta_r^\pi\right) \;=\; 0 \qquad \forall\, a\in\mathcal{A}_+ \tag{EC.11}$$

$$\lambda_a^= + \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\eta_r^\pi - \zeta_r^\pi\right) \;=\; 0 \qquad \forall\, a\in\mathcal{A}_= \tag{EC.12}$$

$$\eta, \zeta, \lambda^-, \lambda^+ \;\geq\; 0 \tag{EC.13}$$

Let $(\hat{\eta}, \hat{\zeta}, \hat{\lambda}^-, \hat{\lambda}^+, \hat{\lambda}^=)$ be an optimal solution of (EC.8)–(EC.13). Let $\breve{q}:\mathbb{R}_+ \mapsto \prod_{\pi\in\Pi}\mathbb{R}^{\hat{\mathcal{R}}^\pi}$ be given by $\breve{q}(\alpha) := \hat{q} + \alpha\left(\hat{\eta} - \hat{\zeta}\right)$. Consider any $\varepsilon \in (0, \hat{z}/\sum_{a\in\mathcal{A}}\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi)$.

For any $a\in\mathcal{A}_0$, it holds that

$$0 \overset{\text{(EC.13)}}{\leq} \alpha\hat{\lambda}_a^+ \overset{\text{(EC.10)}}{=} \alpha\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) \;=\; \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\breve{q}_r^\pi(\alpha) - \hat{q}_r^\pi\right) \;=\; x_a(\breve{q}(\alpha)) - x_a(\hat{q}) \tag{EC.14}$$

Thus $x_a(\breve{q}(\alpha)) \geq x_a(\hat{q}) = 0$ for all $\alpha \geq 0$. Also note that $x_a(\breve{q}(\cdot))$ is continuous, and thus $\lim_{\alpha\downarrow 0} x_a(\breve{q}(\alpha)) = x_a(\hat{q})$ and $\lim_{\alpha\downarrow 0} t_a\left(x_a(\breve{q}(\alpha))\right) \leq t_a^+\left(x_a(\hat{q})\right)$. (It is possible that $\hat{\lambda}_a^+ = 0$ and $x_a(\breve{q}(\alpha)) = x_a(\hat{q})$ for all α, in which case it may hold that $t_a\left(x_a(\breve{q}(\alpha))\right) < t_a^+\left(x_a(\hat{q})\right)$ for all α.) Let $A_a > 0$ be such that $t_a\left(x_a(\breve{q}(\alpha))\right) \leq t_a^+\left(x_a(\hat{q})\right) + \varepsilon$ for all $\alpha\in[0, A_a]$. Then, for all $\alpha\in[0, A_a]$ it holds that

$$\begin{aligned}\hat{\psi}_a\left(\breve{q}(\alpha)\right) - \hat{\psi}_a\left(\hat{q}\right) \;&:=\; \int_0^{x_a(\breve{q}(\alpha))} t_a(y)\,\mathrm{d}y - \int_0^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y \;=\; \int_{x_a(\hat{q})}^{x_a(\breve{q}(\alpha))} t_a(y)\,\mathrm{d}y \\ &\leq\; \left[t_a^+\left(x_a(\hat{q})\right) + \varepsilon\right]\left[x_a(\breve{q}(\alpha)) - x_a(\hat{q})\right] \overset{\text{(EC.14)}}{=} \left[t_a^+\left(x_a(\hat{q})\right) + \varepsilon\right]\alpha\hat{\lambda}_a^+\end{aligned} \tag{EC.15}$$

It follows from (EC.9) and (EC.13) that $\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi \leq 1$ and thus

$$\varepsilon\hat{\lambda}_a^+ \overset{\text{(EC.10)}}{=} \varepsilon\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) \;<\; \hat{z}\frac{\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi}{\sum_{a'\in\mathcal{A}_0\cup\mathcal{A}_+}\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_{a'}^\pi} D^\pi} \tag{EC.16}$$

Consider any $a\in\mathcal{A}_+$. It follows from $t_a^-\left(x_a(\hat{q})\right) < t_a^+\left(x_a(\hat{q})\right)$ and complementary slackness that $\hat{\lambda}_a^-\hat{\lambda}_a^+ = 0$. First consider $a\in\mathcal{A}_{++} := \left\{a\in\mathcal{A}_+ : \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) \geq 0\right\}$. Then it holds that

$$\alpha\left(\hat{\lambda}_a^+ - \hat{\lambda}_a^-\right) \overset{\text{(EC.11)}}{=} \alpha\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) \;=\; \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\breve{q}_r^\pi(\alpha) - \hat{q}_r^\pi\right) \;=\; x_a(\breve{q}(\alpha)) - x_a(\hat{q}) \;\geq\; 0 \tag{EC.17}$$

Thus $\hat{\lambda}_a^- = 0$ and $x_a(\breve{q}(\alpha)) \geq x_a(\hat{q}) \geq 0$ for all $\alpha \geq 0$. Let $A_a > 0$ be such that $t_a\left(x_a(\breve{q}(\alpha))\right) \leq t_a^+\left(x_a(\hat{q})\right) + \varepsilon$ for all $\alpha\in[0, A_a]$. Then, for all $\alpha\in[0, A_a]$ it holds that

$$\hat{\psi}_a\left(\breve{q}(\alpha)\right) - \hat{\psi}_a\left(\hat{q}\right) \;:=\; \int_0^{x_a(\breve{q}(\alpha))} t_a(y)\,\mathrm{d}y - \int_0^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y \;=\; \int_{x_a(\hat{q})}^{x_a(\breve{q}(\alpha))} t_a(y)\,\mathrm{d}y$$

$$\leq \left[t_a^+\left(x_a(\hat{q})\right)+\varepsilon\right]\left[x_a(\check{q}(\alpha))-x_a(\hat{q})\right] \overset{(EC.17)}{=} \left[t_a^+\left(x_a(\hat{q})\right)+\varepsilon\right]\alpha\hat{\lambda}_a^+ \tag{EC.18}$$

It follows from (EC.9) and (EC.13) that $\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi \leq 1$ and thus

$$\varepsilon\hat{\lambda}_a^+ = \varepsilon\left(\hat{\lambda}_a^+ - \hat{\lambda}_a^-\right) \overset{(EC.11)}{=} \varepsilon\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) < \hat{z}\frac{\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi}{\sum_{a'\in\mathcal{A}_0\cup\mathcal{A}_+}\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_{a'}^\pi} D^\pi} \tag{EC.19}$$

Next, consider $a \in \mathcal{A}_{+-} := \left\{a \in \mathcal{A}_+ : \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) < 0\right\}$. Then it holds that

$$\alpha\left(\hat{\lambda}_a^+ - \hat{\lambda}_a^-\right) \overset{(EC.11)}{=} \alpha\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) = \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\check{q}_r^\pi(\alpha) - \hat{q}_r^\pi\right) = x_a(\check{q}(\alpha)) - x_a(\hat{q}) \leq 0 \tag{EC.20}$$

Thus $\hat{\lambda}_a^+ = 0$. Note that $x_a(\hat{q}) > 0$. Also note that $\lim_{\alpha\downarrow 0} x_a(\check{q}(\alpha)) = x_a(\hat{q})$ and $\lim_{\alpha\downarrow 0} t_a\left(x_a(\check{q}(\alpha))\right) \geq t_a^-\left(x_a(\hat{q})\right)$. Let $A_a > 0$ be such that $t_a\left(x_a(\check{q}(\alpha))\right) \geq t_a^-\left(x_a(\hat{q})\right) - \varepsilon$ and $x_a(\check{q}(\alpha)) \geq 0$ for all $\alpha \in [0, A_a]$. Then, for all $\alpha \in [0, A_a]$ it holds that

$$\begin{aligned}\hat{\psi}_a\left(\check{q}(\alpha)\right) - \hat{\psi}_a\left(\hat{q}\right) &:= \int_0^{x_a(\check{q}(\alpha))} t_a(y)\,\mathrm{d}y - \int_0^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y = -\int_{x_a(\check{q}(\alpha))}^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y \\ &\leq \left[t_a^-\left(x_a(\hat{q})\right) - \varepsilon\right]\left[x_a(\check{q}(\alpha)) - x_a(\hat{q})\right] \overset{(EC.20)}{=} -\left[t_a^-\left(x_a(\hat{q})\right) - \varepsilon\right]\alpha\hat{\lambda}_a^-\end{aligned} \tag{EC.21}$$

It follows from (EC.9) and (EC.13) that $\hat{\zeta}_r^\pi - \hat{\eta}_r^\pi \leq 1$ and thus

$$\varepsilon\hat{\lambda}_a^- = \varepsilon\left(\hat{\lambda}_a^- - \hat{\lambda}_a^+\right) \overset{(EC.11)}{=} \varepsilon\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\zeta}_r^\pi - \hat{\eta}_r^\pi\right) < \hat{z}\frac{\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi}{\sum_{a'\in\mathcal{A}_0\cup\mathcal{A}_+}\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_{a'}^\pi} D^\pi} \tag{EC.22}$$

Consider any $a \in \mathcal{A}_=$. First consider $a \in \mathcal{A}_{=+} := \left\{a \in \mathcal{A}_= : \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) \geq 0\right\}$. Then it holds that

$$-\alpha\hat{\lambda}_a^= \overset{(EC.12)}{=} \alpha\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) = \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\check{q}_r^\pi(\alpha) - \hat{q}_r^\pi\right) = x_a(\check{q}(\alpha)) - x_a(\hat{q}) \geq 0 \tag{EC.23}$$

Thus $\hat{\lambda}_a^= \leq 0$ and $x_a(\check{q}(\alpha)) \geq x_a(\hat{q}) \geq 0$ for all $\alpha \geq 0$. Let $A_a > 0$ be such that $t_a\left(x_a(\check{q}(\alpha))\right) \leq t_a\left(x_a(\hat{q})\right) + \varepsilon$ for all $\alpha \in [0, A_a]$. Then, for all $\alpha \in [0, A_a]$ it holds that

$$\begin{aligned}\hat{\psi}_a\left(\check{q}(\alpha)\right) - \hat{\psi}_a\left(\hat{q}\right) &:= \int_0^{x_a(\check{q}(\alpha))} t_a(y)\,\mathrm{d}y - \int_0^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y = \int_{x_a(\hat{q})}^{x_a(\check{q}(\alpha))} t_a(y)\,\mathrm{d}y \\ &\leq \left[t_a\left(x_a(\hat{q})\right) + \varepsilon\right]\left[x_a(\check{q}(\alpha)) - x_a(\hat{q})\right] \overset{(EC.23)}{=} -\left[t_a\left(x_a(\hat{q})\right) + \varepsilon\right]\alpha\hat{\lambda}_a^=\end{aligned} \tag{EC.24}$$

It follows from (EC.9) and (EC.13) that $\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi \leq 1$ and thus

$$-\varepsilon\hat{\lambda}_a^{=} \overset{\text{(EC.12)}}{=} \varepsilon \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) < \hat{z}\frac{\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi}{\sum_{a'\in\mathcal{A}_0\cup\mathcal{A}_+}\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_{a'}^\pi} D^\pi} \tag{EC.25}$$

Next consider $a \in \mathcal{A}_{=-} := \left\{a \in \mathcal{A}_= : \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) < 0\right\}$. Then it holds that

$$-\alpha\hat{\lambda}_a^{=} \overset{\text{(EC.12)}}{=} \alpha\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) = \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\check{q}_r^\pi(\alpha) - \hat{q}_r^\pi\right) = x_a(\check{q}(\alpha)) - x_a(\hat{q}) \leq 0 \tag{EC.26}$$

Thus $\hat{\lambda}_a^{=} \geq 0$. Note that $x_a(\hat{q}) > 0$. Let $A_a > 0$ be such that $t_a\big(x_a(\check{q}(\alpha))\big) \geq t_a\big(x_a(\hat{q})\big) - \varepsilon$ for all $\alpha \in [0, A_a]$. Then, for all $\alpha \in [0, A_a]$ it holds that

$$\begin{aligned}\hat{\psi}_a\big(\check{q}(\alpha)\big) - \hat{\psi}_a\big(\hat{q}\big) &:= \int_0^{x_a(\check{q}(\alpha))} t_a(y)\,\mathrm{d}y - \int_0^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y = -\int_{x_a(\check{q}(\alpha))}^{x_a(\hat{q})} t_a(y)\,\mathrm{d}y \\ &\leq \left[t_a\big(x_a(\hat{q})\big) - \varepsilon\right]\left[x_a(\check{q}(\alpha)) - x_a(\hat{q})\right] \overset{\text{(EC.26)}}{=} -\left[t_a\big(x_a(\hat{q})\big) - \varepsilon\right]\alpha\hat{\lambda}_a^{=}\end{aligned} \tag{EC.27}$$

It follows from (EC.9) and (EC.13) that $\hat{\zeta}_r^\pi - \hat{\eta}_r^\pi \leq 1$ and thus

$$\varepsilon\hat{\lambda}_a^{=} \overset{\text{(EC.12)}}{=} \varepsilon \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi\left(\hat{\zeta}_r^\pi - \hat{\eta}_r^\pi\right) < \hat{z}\frac{\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi}{\sum_{a'\in\mathcal{A}_0\cup\mathcal{A}_+}\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_{a'}^\pi} D^\pi} \tag{EC.28}$$

Let $A := \min\{A_a : a \in \mathcal{A}\} > 0$. Note that $x_a(\check{q}(\alpha)) \geq 0$ for all $\alpha \in [0, A]$ and all $a \in \mathcal{A}$, and thus $\check{q}(\alpha) \in Q$ for all $\alpha \in [0, A]$. Then, for all $\alpha \in (0, A]$ it holds that

$$\begin{aligned}\hat{\psi}\big(\check{q}(\alpha)\big) - \hat{\psi}\big(\hat{q}\big) &= \sum_{a\in\mathcal{A}}\left[\hat{\psi}_a\big(\check{q}(\alpha)\big) - \hat{\psi}_a\big(\hat{q}\big)\right] \\ &= \sum_{a\in\mathcal{A}_0}\left[\hat{\psi}_a\big(\check{q}(\alpha)\big) - \hat{\psi}_a\big(\hat{q}\big)\right] + \sum_{a\in\mathcal{A}_{++}}\left[\hat{\psi}_a\big(\check{q}(\alpha)\big) - \hat{\psi}_a\big(\hat{q}\big)\right] + \sum_{a\in\mathcal{A}_{+-}}\left[\hat{\psi}_a\big(\check{q}(\alpha)\big) - \hat{\psi}_a\big(\hat{q}\big)\right] \\ &\quad + \sum_{a\in\mathcal{A}_{=+}}\left[\hat{\psi}_a\big(\check{q}(\alpha)\big) - \hat{\psi}_a\big(\hat{q}\big)\right] + \sum_{a\in\mathcal{A}_{=-}}\left[\hat{\psi}_a\big(\check{q}(\alpha)\big) - \hat{\psi}_a\big(\hat{q}\big)\right] \\ &\overset{\text{(EC.15)(EC.18)(EC.21)(EC.24)(EC.27)}}{\leq} \sum_{a\in\mathcal{A}_0}\left[t_a^+\big(x_a(\hat{q})\big) + \varepsilon\right]\alpha\hat{\lambda}_a^+ \\ &\quad + \sum_{a\in\mathcal{A}_{++}}\left[t_a^+\big(x_a(\hat{q})\big) + \varepsilon\right]\alpha\hat{\lambda}_a^+ - \sum_{a\in\mathcal{A}_{+-}}\left[t_a^-\big(x_a(\hat{q})\big) - \varepsilon\right]\alpha\hat{\lambda}_a^- \\ &\quad - \sum_{a\in\mathcal{A}_{=+}}\left[t_a\big(x_a(\hat{q})\big) + \varepsilon\right]\alpha\hat{\lambda}_a^{=} - \sum_{a\in\mathcal{A}_{=-}}\left[t_a\big(x_a(\hat{q})\big) - \varepsilon\right]\alpha\hat{\lambda}_a^{=} \\ &\overset{\text{(EC.16)(EC.19)(EC.22)(EC.25)(EC.28)}}{<} \alpha\sum_{a\in\mathcal{A}_0} t_a^+\big(x_a(\hat{q})\big)\hat{\lambda}_a^+\end{aligned}$$

$$
\begin{aligned}
&\quad + \alpha \sum_{a\in\mathcal{A}_{++}} t_a^+\big(x_a(\hat{q})\big)\hat{\lambda}_a^+ - \alpha \sum_{a\in\mathcal{A}_{+-}} t_a^-\big(x_a(\hat{q})\big)\hat{\lambda}_a^- \\
&\quad - \alpha \sum_{a\in\mathcal{A}_{=}} t_a\big(x_a(\hat{q})\big)\hat{\lambda}_a^= + \alpha \sum_{a\in\mathcal{A}} \hat{z}\frac{\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_a^\pi} D^\pi}{\sum_{a'\in\mathcal{A}}\sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}_{a'}^\pi} D^\pi} \\
&= \alpha \sum_{a\in\mathcal{A}_0} t_a^+\big(x_a(\hat{q})\big)\hat{\lambda}_a^+ + \alpha \sum_{a\in\mathcal{A}_+} t_a^+\big(x_a(\hat{q})\big)\hat{\lambda}_a^+ - \alpha \sum_{a\in\mathcal{A}_+} t_a^-\big(x_a(\hat{q})\big)\hat{\lambda}_a^- - \alpha \sum_{a\in\mathcal{A}_=} t_a\big(x_a(\hat{q})\big)\hat{\lambda}_a^= + \alpha\hat{z} \\
&= \alpha \sum_{a\in\mathcal{A}_0\cup\mathcal{A}_+} t_a^+\big(x_a(\hat{q})\big)\hat{\lambda}_a^+ - \alpha \sum_{a\in\mathcal{A}_+} t_a^-\big(x_a(\hat{q})\big)\hat{\lambda}_a^- - \alpha \sum_{a\in\mathcal{A}_=} t_a\big(x_a(\hat{q})\big)\hat{\lambda}_a^= \\
&\quad + \alpha \left\{ \sum_{\pi\in\Pi}\sum_{r\in\hat{\mathcal{R}}^\pi} \hat{g}_r^\pi \left(\hat{\eta}_r^\pi - \hat{\zeta}_r^\pi\right) + \sum_{a\in\mathcal{A}_+} t_a^-(x_a(\hat{q}))\hat{\lambda}_a^- - \sum_{a\in\mathcal{A}_0\cup\mathcal{A}_+} t_a^+(x_a(\hat{q}))\hat{\lambda}_a^+ + \sum_{a\in\mathcal{A}_=} t_a(x_a(\hat{q}))\hat{\lambda}_a^= \right\} \\
&= \alpha\, \hat{g}^\top\big(\hat{\eta} - \hat{\zeta}\big) \;=\; \hat{g}^\top\big(\breve{q}(\alpha) - \hat{q}\big)
\end{aligned}
$$

and thus $\hat{g} \notin \partial\hat{\psi}(\hat{q})$. Therefore, it has been established that

$$
\partial\hat{\psi}(q) \;=\; \left\{ g = \left(g_r^\pi, \pi\in\Pi, r\in\hat{\mathcal{R}}^\pi\right) \,:\, g_r^\pi = D^\pi \sum_{a\in\mathcal{A}_r^\pi} \tau_a,\ \tau_a \in \left[t_a^-(x_a(q)), t_a^+(x_a(q))\right]\ \forall\, a\in\mathcal{A} \right\}
$$

and the result of the lemma follows. □

EC.2. Proof for Result in Section 4.2

Lemma 6 *A vector $q^* \in \prod_{\pi\in\Pi}\Delta_{\hat{\mathcal{R}}^\pi}$ is optimal for problem* (18) *if and only if there exist $\tau^* = (\tau_a^*, a\in\mathcal{A})$ with $\tau_a^* \in \left[t_a^-(x_a(q^*)), t_a^+(x_a(q^*))\right]$, $w^* = \left(w_r^{\pi*}, \pi\in\Pi, r\in\hat{\mathcal{R}}^\pi\right)$ with $w_r^{\pi*} \in (G^\pi)^{-1}\left(L_r^\pi(q^{\pi*})\right)$, and $\lambda^* = (\lambda^{\pi*}, \pi\in\Pi)$, such that the following holds:*

$$
\begin{aligned}
&q_r^{\pi*}\left\{ D^\pi\left[\sum_{a\in\mathcal{A}_r^\pi}\tau_a^* + \sum_{\{r'\in\hat{\mathcal{R}}^\pi\,:\, r\preceq r', r'+1\in\hat{\mathcal{R}}^\pi\}} \left(S_{r'}^\pi - S_{r'+1}^\pi\right) w_{r'}^{\pi*}\right] + \lambda^{\pi*}\right\} \;=\; 0 \qquad \forall\,\pi\in\Pi,\ \forall\, r\in\hat{\mathcal{R}}^\pi \\
&D^\pi\left[\sum_{a\in\mathcal{A}_r^\pi}\tau_a^* + \sum_{\{r'\in\hat{\mathcal{R}}^\pi\,:\, r\preceq r', r'+1\in\hat{\mathcal{R}}^\pi\}} \left(S_{r'}^\pi - S_{r'+1}^\pi\right) w_{r'}^{\pi*}\right] + \lambda^{\pi*} \;\geq\; 0 \qquad \forall\,\pi\in\Pi,\ \forall\, r\in\hat{\mathcal{R}}^\pi
\end{aligned}
$$

Proof. Convexity of $\Upsilon : \prod_{\pi\in\Pi}\Delta_{\hat{\mathcal{R}}^\pi} \mapsto \mathbb{R}$ follows from Lemmas 3 and 4. It follows from convexity of Υ and $\prod_{\pi\in\Pi}\Delta_{\hat{\mathcal{R}}^\pi}$ that $q^* \in \prod_{\pi\in\Pi}\Delta_{\hat{\mathcal{R}}^\pi}$ is optimal for problem (18) if and only if $0 \in \partial\Upsilon(q^*)$. Note that $\Upsilon(q) = \psi(q) + \phi(q) + \sum_{\pi\in\Pi}\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$ and

$$
\mathrm{ri}\left(\mathrm{dom}(\psi)\right) \cap \mathrm{ri}\left(\mathrm{dom}(\phi)\right) \cap \mathrm{ri}\left(\mathrm{dom}\left(\sum_{\pi\in\Pi}\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}\right)\right) \;=\; \prod_{\pi\in\Pi}\Delta^\circ_{\hat{\mathcal{R}}^\pi} \;\neq\; \varnothing
$$

and thus it follows from the Moreau-Rockafellar Theorem (Rockafellar 1997, Theorem 23.8) that $\partial\Upsilon(q) = \partial\psi(q) + \partial\phi(q) + \partial\sum_{\pi\in\Pi}\delta^\pi_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^\pi)$. Hence, it follows from Lemmas 3 and 4 that

$0 \in \partial\Upsilon(q^*)$ if and only if there exist $\tau^* = (\tau_a^*, a \in \mathcal{A})$ with $\tau_a^* \in \left[t_a^-(x_a(q^*)), t_a^+(x_a(q^*))\right]$, $w^* = \left(w_r^{\pi*}, \pi \in \Pi, r \in \hat{\mathcal{R}}^\pi\right)$ with $w_r^{\pi*} \in (G^\pi)^{-1}\left(L_r^\pi(q^{\pi*})\right)$, and $\nu^{\pi*} = (\nu_r^{\pi*}, r \in \hat{\mathcal{R}}^\pi) \in \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^{\pi*})$ for all $\pi \in \Pi$, such that

$$0 \;=\; D^\pi \left[\sum_{a \in \mathcal{A}_r^\pi} \tau_a^* + \sum_{\{r' \in \hat{\mathcal{R}}^\pi \,:\, r \preceq r', r'+1 \in \hat{\mathcal{R}}^\pi\}} \left(S_{r'}^\pi - S_{r'+1}^\pi\right) w_{r'}^{\pi*} \right] + \nu_r^{\pi*} \qquad \forall\, \pi \in \Pi,\ \forall\, r \in \hat{\mathcal{R}}^\pi \quad \text{(EC.29)}$$

Note that $\nu^{\pi*} \in \mathcal{N}_{\Delta_{\hat{\mathcal{R}}^\pi}}(q^{\pi*})$ if and only if there are $\lambda^{\pi*}$ and $\mu_r^{\pi*} \geq 0$ for all $r \in \hat{\mathcal{R}}^\pi$, such that $\nu_r^{\pi*} = \lambda^{\pi*} - \mu_r^{\pi*}$ and $\mu_r^{\pi*} q_r^{\pi*} = 0$ for all $r \in \hat{\mathcal{R}}^\pi$. Therefore, in both the cases $q_r^{\pi*} = 0$ and $q_r^{\pi*} > 0$, (EC.29) is equivalent to (19)–(20). □

EC.3. Proofs for Results in Section 4.3

Lemma 8 *Assume that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous for all $a \in \mathcal{A}$. Consider any optimal solution q for problem* (18)*, and the resulting output w and $\mathcal{R}_{\min}^\pi$, $\pi \in \Pi$, of Algorithm 1. Then the following hold:*

(1) The thresholds w satisfy $\underline{w}^\pi = w_0^\pi \leq w_1^\pi \leq \cdots \leq w_{\hat{R}^\pi}^\pi = \overline{w}^\pi$ for all $\pi \in \Pi$.

(2) Every path with positive flow is an efficient path, that is, $\left\{r \in \hat{\mathcal{R}}^\pi \,:\, q_r^\pi > 0\right\} \subset \mathcal{R}_{\min}^\pi$.

(3) The thresholds w satisfy $w_r^\pi \in (G^\pi)^{-1}\left(L_r^\pi(q^\pi)\right)$ for all $\pi \in \Pi$, $r \in \hat{\mathcal{R}}^\pi$.

(4) For every $\pi \in \Pi$ and every $r_\ell \in \mathcal{R}_{\min}^\pi$ it holds that

$$T_{r_\ell}^\pi(x(q)) + w S_{r_\ell}^\pi \;\leq\; T_r^\pi(x(q)) + w S_r^\pi \qquad \forall\, r \in \hat{\mathcal{R}}^\pi,\ \forall\, w \in [w_{r_{\ell-1}}^\pi, w_{r_\ell}^\pi]$$

where $w_{r_0}^\pi = w_0^\pi = \underline{w}^\pi$.

Proof. Consider any $\pi \in \Pi$. Let $\hat{\mathcal{R}}^\pi$ be indexed with $r = 1, \ldots, \hat{R}^\pi$, let $\mathcal{R}_{\min}^\pi = \{r_1, \ldots, r_L\}$, and let $p_r^\pi = L_r^\pi(q^\pi)$. All the properties will be established together by induction.

Initialization.

(1) Note that $\underline{w}^\pi = w_0^\pi = \cdots = w_{r_1-1}^\pi$, and $w_{r_\ell}^\pi = w_{r_\ell+1}^\pi = \cdots = w_{r_{\ell+1}-1}^\pi$ for $\ell = 1, \ldots, L-1$, and $w_{r_L}^\pi = w_{r_L+1}^\pi = \cdots = w_{\hat{R}^\pi}^\pi = \overline{w}^\pi$. Thus, to establish (1), it remains to show that $\underline{w}^\pi \leq w_{r_\ell}^\pi \leq w_{r_{\ell+1}}^\pi \leq \overline{w}^\pi$ for $\ell = 1, \ldots, L-1$. Next we show that $\underline{w}^\pi \leq w_{r_1}^\pi$. In Algorithm 1, $w_{r_1}^\pi$ is determined by (24) or (26) or (28) or (30). First consider the case in which $w_{r_1}^\pi$ is determined by (24) or (28). If $w_1^{\min} > \overline{w}^\pi$, then $L = 1$ and $\underline{w}^\pi \leq w_{r_1}^\pi = \overline{w}^\pi$. Otherwise, $w_{r_1}^\pi = w_1^{\min} \leq \overline{w}^\pi$, $S_{r_1}^\pi > S_{r_2}^\pi$, and $T_{r_1}^\pi(x(q)) + w_{r_1}^\pi S_{r_1}^\pi = T_{r_2}^\pi(x(q)) + w_{r_1}^\pi S_{r_2}^\pi$. Suppose that $\underline{w}^\pi > w_{r_1}^\pi$. Then

$$\left(S_{r_1}^\pi - S_{r_2}^\pi\right)\underline{w}^\pi \;>\; \left(S_{r_1}^\pi - S_{r_2}^\pi\right) w_{r_1}^\pi \;=\; T_{r_2}^\pi(x(q)) - T_{r_1}^\pi(x(q)) \quad \Rightarrow \quad T_{r_1}^\pi(x(q)) + \underline{w}^\pi S_{r_1}^\pi \;>\; T_{r_2}^\pi(x(q)) + \underline{w}^\pi S_{r_2}^\pi$$

which contradicts (23). Thus $\underline{w}^\pi \leq w_{r_1}^\pi \leq \overline{w}^\pi$. If $w_{r_1}^\pi$ is determined by (26) or (30), then it follows immediately that $\underline{w}^\pi \leq w_{r_1}^\pi \leq \overline{w}^\pi$.

(2) We will show that for any $r \in \hat{\mathcal{R}}^\pi \setminus \mathcal{R}^\pi_{\min}$ it holds that $q^\pi_r = 0$. First consider any $r < r_1$. Then $S^\pi_r \geq S^\pi_{r_1}$ and it follows from (23) that $T^\pi_r(x(q)) + \underline{w}^\pi S^\pi_r > T^\pi_{r_1}(x(q)) + \underline{w}^\pi S^\pi_{r_1}$. Suppose that $q^\pi_r > 0$. Then for all $r' \geq r$ it holds that $p^\pi_{r'} \geq q^\pi_r > 0$, and thus any $\hat{w} \in (G^\pi)^{-1}\left(p^\pi_{r'}\right)$ satisfies $\hat{w} \geq \underline{w}^\pi$. Consider the directional derivative of Υ at q in the feasible direction $e^\pi_{r_1} - e^\pi_r$:

$$\begin{aligned}
\max \Bigg\{ & \left(D^\pi \left[T^\pi_{r_1}(x(q)) + \sum_{r'=r_1}^{\hat{R}^\pi - 1} \left(S^\pi_{r'} - S^\pi_{r'+1} \right) \hat{w}^\pi_{r'} \right] + \lambda^\pi - \mu^\pi_{r_1} \right) \\
& - \left(D^\pi \left[T^\pi_r(x(q)) + \sum_{r'=r}^{\hat{R}^\pi - 1} \left(S^\pi_{r'} - S^\pi_{r'+1} \right) \hat{w}^\pi_{r'} \right] + \lambda^\pi \right) : \hat{w}^\pi_1 \in (G^\pi)^{-1}\left(p^\pi_1\right), \ldots, \hat{w}^\pi_{\hat{R}^\pi - 1} \in (G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi - 1}\right), \\
& \qquad \mu^\pi_{r_1} \geq 0,\ \mu^\pi_{r_1} q^\pi_{r_1} = 0 \Bigg\} \\
= \ & \max \left\{ D^\pi \left[T^\pi_{r_1}(x(q)) - T^\pi_r(x(q)) - \sum_{r'=r}^{r_1 - 1} \left(S^\pi_{r'} - S^\pi_{r'+1} \right) \hat{w}^\pi_{r'} \right] : \hat{w}^\pi_r \in (G^\pi)^{-1}\left(p^\pi_r\right), \ldots, \hat{w}^\pi_{r_1 - 1} \in (G^\pi)^{-1}\left(p^\pi_{r_1 - 1}\right) \right\} \\
\leq \ & D^\pi \left[T^\pi_{r_1}(x(q)) - T^\pi_r(x(q)) - \sum_{r'=r}^{r_1 - 1} \left(S^\pi_{r'} - S^\pi_{r'+1} \right) \underline{w}^\pi \right] \\
= \ & D^\pi \left[T^\pi_{r_1}(x(q)) - T^\pi_r(x(q)) + \left(S^\pi_{r_1} - S^\pi_r \right) \underline{w}^\pi \right] \ < \ 0
\end{aligned}$$

which contradicts the optimality of q. Thus for all $r < r_1$, it holds that $q^\pi_r = 0$.

(3) Note that $p^\pi_0 = 0$ and $w^\pi_0 = \underline{w}^\pi \in (G^\pi)^{-1}(0)$, and thus $w^\pi_0 \in (G^\pi)^{-1}\left(p^\pi_0\right)$. By Algorithm 1, $w^\pi_r = \underline{w}^\pi$ for all $r < r_1$. It was established in (2) above that $q^\pi_r = 0$ for all $r < r_1$, and thus $p^\pi_r = 0$ and $w^\pi_r \in (G^\pi)^{-1}\left(p^\pi_r\right)$ for all $r < r_1$. Also, $p^\pi_{\hat{R}^\pi} = 1$ and $w^\pi_{\hat{R}^\pi} = \overline{w}^\pi \in (G^\pi)^{-1}(1)$, and thus $w^\pi_{\hat{R}^\pi} \in (G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi}\right)$.

Next we show by contradiction that $w^\pi_{r_1} \in (G^\pi)^{-1}\left(p^\pi_{r_1}\right)$. For each $r \in \{0\} \cup \hat{\mathcal{R}}^\pi$, if $p^\pi_r \in (0,1)$ then let $(G^\pi)^{-1}\left(p^\pi_r\right) = [\underline{w}^\pi_r, \overline{w}^\pi_r]$, if $p^\pi_r = 0$ then let $(G^\pi)^{-1}\left(p^\pi_r\right) = (-\infty, \overline{w}^\pi_r]$, and if $p^\pi_r = 1$ then let $(G^\pi)^{-1}\left(p^\pi_r\right) = [\underline{w}^\pi_r, \infty)$.

Case 1: Suppose that $w^\pi_{r_1} < \underline{w}^\pi_{r_1}$.

Note that $w^\pi_{r_1} < \underline{w}^\pi_{r_1}$ implies that $G^\pi(w^\pi_{r_1}) < p^\pi_{r_1}$. Also, it follows from $q^\pi_r = 0$ for all $r < r_1$ and $p^\pi_{r_1} > 0$ that $q^\pi_{r_1} > 0$. In Algorithm 1, $w^\pi_{r_1}$ is determined by (24) or (26) or (28) or (30). It follows from $w^\pi_{r_1} < \underline{w}^\pi_{r_1}$ that $w^\pi_{r_1}$ is not determined by (26). Also, it follows from $w^\pi_{r_1} < \underline{w}^\pi_{r_1} \leq \overline{w}^\pi$ that $w^\pi_{r_1}$ is not determined by (30). Thus $w^\pi_{r_1}$ is determined by (24) or (28). In both cases, $S^\pi_{r_1} > S^\pi_{r_2}$, and $w^\pi_r = w^\pi_{r_1}$ for all $r \in \{r_1, \ldots, r_2 - 1\}$. It follows from $w^\pi_{r_1} < \underline{w}^\pi_{r_1} \leq \overline{w}^\pi$ that

$$w^\pi_{r_1} \ = \ w^{\min}_1 \ = \ \min \left\{ \frac{T^\pi_r(x(q)) - T^\pi_{r_1}(x(q))}{S^\pi_{r_1} - S^\pi_r} \ : \ S^\pi_{r_1} > S^\pi_r,\ r \in \hat{\mathcal{R}}^\pi \right\} \ = \ \frac{T^\pi_{r_2}(x(q)) - T^\pi_{r_1}(x(q))}{S^\pi_{r_1} - S^\pi_{r_2}}$$

$$\Rightarrow \quad T^\pi_{r_1}(x(q)) + w^\pi_{r_1} S^\pi_{r_1} \;=\; T^\pi_{r_2}(x(q)) + w^\pi_{r_1} S^\pi_{r_2}$$

$$\Rightarrow \quad T^\pi_{r_1}(x(q)) + w S^\pi_{r_1} \;>\; T^\pi_{r_2}(x(q)) + w S^\pi_{r_2} \quad \forall\, w > w^\pi_{r_1} \tag{EC.30}$$

It follows from $q^\pi_{r_1} > 0$ that direction $e^\pi_{r_2} - e^\pi_{r_1}$ is feasible. Next, consider the directional derivative of Υ at q in the direction $e^\pi_{r_2} - e^\pi_{r_1}$:

$$\begin{aligned}
&\max \left\{ \left(D^\pi \left[T^\pi_{r_2}(x(q)) + \sum_{r=r_2}^{\hat{R}^\pi - 1} \left(S^\pi_r - S^\pi_{r+1} \right) \hat{w}^\pi_r \right] + \lambda^\pi - \mu^\pi_{r_2} \right) \right. \\
&\qquad - \left(D^\pi \left[T^\pi_{r_1}(x(q)) + \sum_{r=r_1}^{\hat{R}^\pi - 1} \left(S^\pi_r - S^\pi_{r+1} \right) \hat{w}^\pi_r \right] + \lambda^\pi \right) : \; \hat{w}^\pi_1 \in (G^\pi)^{-1}\left(p^\pi_1\right), \ldots, \hat{w}^\pi_{\hat{R}^\pi - 1} \in (G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi - 1}\right), \\
&\qquad\qquad \left. \mu^\pi_{r_2} \geq 0,\; \mu^\pi_{r_2} q^\pi_{r_2} = 0 \right\} \\
&= \max \left\{ D^\pi \left[T^\pi_{r_2}(x(q)) - T^\pi_{r_1}(x(q)) - \sum_{r=r_1}^{r_2 - 1} \left(S^\pi_r - S^\pi_{r+1} \right) \hat{w}^\pi_r \right] : \; \hat{w}^\pi_{r_1} \in (G^\pi)^{-1}\left(p^\pi_{r_1}\right), \ldots, \hat{w}^\pi_{r_2 - 1} \in (G^\pi)^{-1}\left(p^\pi_{r_2 - 1}\right) \right\} \\
&\leq D^\pi \left[T^\pi_{r_2}(x(q)) - T^\pi_{r_1}(x(q)) - \sum_{r=r_1}^{r_2 - 1} \left(S^\pi_r - S^\pi_{r+1} \right) \underline{w}^\pi_{r_1} \right] \\
&= D^\pi \left[T^\pi_{r_2}(x(q)) - T^\pi_{r_1}(x(q)) + \left(S^\pi_{r_2} - S^\pi_{r_1} \right) \underline{w}^\pi_{r_1} \right] \overset{\text{(EC.30)}}{<} 0
\end{aligned}$$

which contradicts the optimality of q.

Case 2: Suppose that $w^\pi_{r_1} > \overline{w}^\pi_{r_1}$.

Note that $w^\pi_{r_1} > \overline{w}^\pi_{r_1}$ implies that $1 \geq G^\pi(w^\pi_{r_1}) > p^\pi_{r_1}$. It follows from $p^\pi_{r_1} < 1$ that $\left\{ r \in \{r_1 + 1, \ldots, \hat{R}^\pi\} : q^\pi_r > 0 \right\} \neq \emptyset$. Let $\check{r} := \min \left\{ r \in \{r_1 + 1, \ldots, \hat{R}^\pi\} : q^\pi_r > 0 \right\}$. It follows that $p^\pi_r = p^\pi_{r_1}$, and $(G^\pi)^{-1}\left(p^\pi_r\right) = (G^\pi)^{-1}\left(p^\pi_{r_1}\right) = [\underline{w}^\pi_{r_1}, \overline{w}^\pi_{r_1}]$ for all $r \in \{r_1, \ldots, \check{r} - 1\}$.

Case 2.1: $S^\pi_{r_1} = S^\pi_{\check{r}}$.

Then $w^\pi_{r_1}$ is determined by (26) or (28) or (30). It follows from $w^\pi_{r_1 - 1} = \underline{w}^\pi \leq \underline{w}^\pi_{r_1} \leq \overline{w}^\pi_{r_1} < w^\pi_{r_1}$ that $w^\pi_{r_1}$ is not determined by (26). If $w^\pi_{r_1}$ is determined by (28) or (30), then $T^\pi_{\check{r}}(x(q)) \geq T^{\min}_1 > T^\pi_{r_1}(x(q))$, and thus $T^\pi_{\check{r}}(x(q)) + w S^\pi_{\check{r}} > T^\pi_{r_1}(x(q)) + w S^\pi_{r_1}$ for all w, which contradicts the optimality of q with $q^\pi_{\check{r}} > 0$.

Case 2.2: $S^\pi_{r_1} > S^\pi_{\check{r}}$.

Then $w^\pi_{r_1}$ is determined by (24) or (26) or (28). It follows from $w^\pi_{r_1 - 1} = \underline{w}^\pi \leq \underline{w}^\pi_{r_1} \leq \overline{w}^\pi_{r_1} < w^\pi_{r_1}$ that $w^\pi_{r_1}$ is not determined by (26). If $w^\pi_{r_1}$ is determined by (24) or (28), then

$$w^\pi_{r_1} \;\leq\; w^{\min}_1 \;=\; \min \left\{ \frac{T^\pi_r(x(q)) - T^\pi_{r_1}(x(q))}{S^\pi_{r_1} - S^\pi_r} \;:\; S^\pi_{r_1} > S^\pi_r,\; r \in \hat{\mathcal{R}}^\pi \right\} \;\leq\; \frac{T^\pi_{\check{r}}(x(q)) - T^\pi_{r_1}(x(q))}{S^\pi_{r_1} - S^\pi_{\check{r}}}$$

$$\Rightarrow \quad T^\pi_{r_1}(x(q)) + w^\pi_{r_1} S^\pi_{r_1} \;\leq\; T^\pi_{\check{r}}(x(q)) + w^\pi_{r_1} S^\pi_{\check{r}}$$

$$\Rightarrow \quad T^\pi_{r_1}(x(q)) + w S^\pi_{r_1} \;<\; T^\pi_{\check{r}}(x(q)) + w S^\pi_{\check{r}} \quad \forall\, w < w^\pi_{r_1} \tag{EC.31}$$

It follows from $q^\pi_{\check{r}} > 0$ that direction $e^\pi_{r_1} - e^\pi_{\check{r}}$ is feasible. Next, consider the directional derivative of Υ at q in the direction $e^\pi_{r_1} - e^\pi_{\check{r}}$:

$$\begin{aligned}
&\max \left\{ \left(D^\pi \left[T^\pi_{r_1}(x(q)) + \sum_{r=r_1}^{\hat{R}^\pi - 1} \left(S^\pi_r - S^\pi_{r+1} \right) \hat{w}^\pi_r \right] + \lambda^\pi - \mu^\pi_{r_1} \right) \right. \\
&\qquad - \left(D^\pi \left[T^\pi_{\check{r}}(x(q)) + \sum_{r=\check{r}}^{\hat{R}^\pi - 1} \left(S^\pi_r - S^\pi_{r+1} \right) \hat{w}^\pi_r \right] + \lambda^\pi \right) : \; \hat{w}^\pi_1 \in (G^\pi)^{-1}\left(p^\pi_1\right), \ldots, \hat{w}^\pi_{\hat{R}^\pi - 1} \in (G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi - 1}\right), \\
&\qquad\qquad \left. \mu^\pi_{r_1} \geq 0,\; \mu^\pi_{r_1} q^\pi_{r_1} = 0 \right\} \\
&= \max \left\{ D^\pi \left[T^\pi_{r_1}(x(q)) - T^\pi_{\check{r}}(x(q)) + \sum_{r=r_1}^{\check{r}-1} \left(S^\pi_r - S^\pi_{r+1} \right) \hat{w}^\pi_r \right] : \; \hat{w}^\pi_{r_1} \in (G^\pi)^{-1}\left(p^\pi_{r_1}\right), \ldots, \hat{w}^\pi_{\check{r}-1} \in (G^\pi)^{-1}\left(p^\pi_{\check{r}-1}\right) \right\} \\
&= D^\pi \left[T^\pi_{r_1}(x(q)) - T^\pi_{\check{r}}(x(q)) + \sum_{r=r_1}^{\check{r}-1} \left(S^\pi_r - S^\pi_{r+1} \right) \overline{w}^\pi_{r_1} \right] \\
&= D^\pi \left[T^\pi_{r_1}(x(q)) - T^\pi_{\check{r}}(x(q)) + \left(S^\pi_{r_1} - S^\pi_{\check{r}} \right) \overline{w}^\pi_{r_1} \right] \overset{\text{(EC.31)}}{<} 0
\end{aligned}$$

which contradicts the optimality of q.

(4) It follows from (23) that

$$T^\pi_{r_1}(x(q)) + \underline{w}^\pi S^\pi_{r_1} \;\leq\; T^\pi_r(x(q)) + \underline{w}^\pi S^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi$$

Thus, for any $w \geq \underline{w}^\pi$ and any $r < r_1$ it holds that $S^\pi_r \geq S^\pi_{r_1}$, and hence

$$\left(S^\pi_r - S^\pi_{r_1} \right) w \;\geq\; \left(S^\pi_r - S^\pi_{r_1} \right) \underline{w}^\pi \qquad \Rightarrow \quad T^\pi_r(x(q)) + w S^\pi_r \overset{(23)}{\geq} T^\pi_{r_1}(x(q)) + w S^\pi_{r_1} \tag{EC.32}$$

In Algorithm 1, $w^\pi_{r_1}$ is determined by (24) or (26) or (28) or (30).

Case 1: $w^\pi_{r_1}$ is determined by (24) or (28):

Then, for any $w \leq w^{\min}_1$ and any $r > r_1$ it holds that either $S^\pi_r = S^\pi_{r_1}$ and $T^\pi_r(x(q)) > T^\pi_{r_1}(x(q))$, or $S^\pi_{r_1} > S^\pi_r$ and

$$w \;\leq\; w^{\min}_1 \;\leq\; \frac{T^\pi_r(x(q)) - T^\pi_{r_1}(x(q))}{S^\pi_{r_1} - S^\pi_r} \qquad \Rightarrow \quad T^\pi_{r_1}(x(q)) + w S^\pi_{r_1} \;\leq\; T^\pi_r(x(q)) + w S^\pi_r \tag{EC.33}$$

It follows from (EC.32) and (EC.33) that

$$T^\pi_{r_1}(x(q)) + wS^\pi_{r_1} \;\leq\; T^\pi_r(x(q)) + wS^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi,\ \forall\, w \in [w^\pi_{r_0}, w^\pi_{r_1}] \subset [\underline{w}^\pi, w^{\min}_1]$$

Case 2: $w^\pi_{r_1}$ is determined by (26):

Case 2.1: $q^\pi_{r_1} = 0$: It was established above that $q^\pi_r = 0$ for all $r < r_1$, and thus $p^\pi_{r_1} = 0$ and $w^\pi_{r_1} = \underline{w}^\pi = \max\left\{w^\pi_{r_0}, \min\left\{w \in (G^\pi)^{-1}\left(p^\pi_{r_1}\right)\right\}\right\}$. It follows from (23) that

$$T^\pi_{r_1}(x(q)) + wS^\pi_{r_1} \;\leq\; T^\pi_r(x(q)) + wS^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi,\ \forall\, w \in [w^\pi_{r_0}, w^\pi_{r_1}] = [\underline{w}^\pi, \underline{w}^\pi]$$

Case 2.2: $q^\pi_{r_1} > 0$: For any $r \in \hat{\mathcal{R}}^\pi$, if $S^\pi_r = S^\pi_{r_1}$, then $T^\pi_r(x(q)) \geq T^\pi_{r_1}(x(q))$ and thus $T^\pi_r(x(q)) + wS^\pi_r \geq T^\pi_{r_1}(x(q)) + wS^\pi_{r_1}$ for all w. Suppose that $w^\pi_{r_1} = \min\left\{w \in (G^\pi)^{-1}\left(p^\pi_{r_1}\right)\right\} > \underline{w}^\pi$ and there is $\hat{r} > r_1$ such that $S^\pi_{\hat{r}} < S^\pi_{r_1}$ and $T^\pi_{r_1}(x(q)) + w^\pi_{r_1}S^\pi_{r_1} > T^\pi_{\hat{r}}(x(q)) + w^\pi_{r_1}S^\pi_{\hat{r}}$. Then consider the directional derivative of Υ at q in the feasible direction $e^\pi_{\hat{r}} - e^\pi_{r_1}$:

$$\begin{aligned}
&\max\left\{\left(D^\pi\left[T^\pi_{\hat{r}}(x(q)) + \sum_{r=\hat{r}}^{\hat{R}^\pi-1}\left(S^\pi_r - S^\pi_{r+1}\right)\hat{w}^\pi_r\right] + \lambda^\pi - \mu^\pi_{\hat{r}}\right)\right.\\
&\qquad -\left(D^\pi\left[T^\pi_{r_1}(x(q)) + \sum_{r=r_1}^{\hat{R}^\pi-1}\left(S^\pi_r - S^\pi_{r+1}\right)\hat{w}^\pi_r\right] + \lambda^\pi\right) :\ \hat{w}^\pi_1 \in (G^\pi)^{-1}\left(p^\pi_1\right), \ldots, \hat{w}^\pi_{\hat{R}^\pi-1} \in (G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi-1}\right),\\
&\qquad\qquad \left.\mu^\pi_{\hat{r}} \geq 0,\ \mu^\pi_{\hat{r}} q^\pi_{\hat{r}} = 0\right\}\\
&=\ \max\left\{D^\pi\left[T^\pi_{\hat{r}}(x(q)) - T^\pi_{r_1}(x(q)) - \sum_{r=r_1}^{\hat{r}-1}\left(S^\pi_r - S^\pi_{r+1}\right)\hat{w}^\pi_r\right] :\ \hat{w}^\pi_{r_1} \in (G^\pi)^{-1}\left(p^\pi_{r_1}\right), \ldots, \hat{w}^\pi_{\hat{r}-1} \in (G^\pi)^{-1}\left(p^\pi_{\hat{r}-1}\right)\right\}\\
&\leq\ D^\pi\left[T^\pi_{\hat{r}}(x(q)) - T^\pi_{r_1}(x(q)) - \sum_{r=r_1}^{\hat{r}-1}\left(S^\pi_r - S^\pi_{r+1}\right)w^\pi_{r_1}\right]\\
&=\ D^\pi\left[T^\pi_{\hat{r}}(x(q)) - T^\pi_{r_1}(x(q)) + w^\pi_{r_1}\left(S^\pi_{\hat{r}} - S^\pi_{r_1}\right)\right] \;<\; 0
\end{aligned}$$

which contradicts the optimality of q. (The first inequality follows from $w^\pi_{r_1} = \min\left\{w \in (G^\pi)^{-1}\left(p^\pi_{r_1}\right)\right\}$.) Thus, for all $r \in \hat{\mathcal{R}}^\pi$ such that $r > r_1$ it holds that

$$\begin{aligned}
&T^\pi_{r_1}(x(q)) + w^\pi_{r_1}S^\pi_{r_1} \;\leq\; T^\pi_r(x(q)) + w^\pi_{r_1}S^\pi_r\\
\Rightarrow\qquad &T^\pi_{r_1}(x(q)) + wS^\pi_{r_1} \;\leq\; T^\pi_r(x(q)) + wS^\pi_r \qquad \forall\, w \leq w^\pi_{r_1}
\end{aligned} \tag{EC.34}$$

It follows from (EC.32) and (EC.34) that

$$T^\pi_{r_1}(x(q)) + wS^\pi_{r_1} \;\leq\; T^\pi_r(x(q)) + wS^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi,\ \forall\, w \in [w^\pi_{r_0}, w^\pi_{r_1}]$$

Case 3: $w^\pi_{r_1}$ is determined by (30).

Then $S^\pi_r \geq S^\pi_{r_1}$ for all $r \in \hat{\mathcal{R}}^\pi$, and thus it follows from (EC.32) that

$$T^\pi_{r_1}(x(q)) + wS^\pi_{r_1} \;\leq\; T^\pi_r(x(q)) + wS^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi,\ \forall\, w \in [w^\pi_{r_0}, w^\pi_{r_1}] \subset [\underline{w}^\pi, \infty)$$

Induction.

Suppose that the following induction hypothesis holds for some $\ell \in \{1, \ldots, L-1\}$:

(1) The thresholds w satisfy $\underline{w}^\pi = w^\pi_0 \leq \cdots \leq w^\pi_{r_{\ell-1}} \leq w^\pi_{r_\ell} \leq \overline{w}^\pi$.

(2) For every $r \in \hat{\mathcal{R}}^\pi \setminus \mathcal{R}^\pi_{\min}$ such that $r < r_\ell$, it holds that $q^\pi_r = 0$.

(3) For every $r \in \hat{\mathcal{R}}^\pi$ such that $r \leq r_\ell$, it holds that $w^\pi_r \in (G^\pi)^{-1}\left(p^\pi_r\right)$.

(4) For every $\ell' \in \{1, \ldots, \ell\}$, it holds that $T^\pi_{r_{\ell'}}(x(q)) + wS^\pi_{r_{\ell'}} \leq T^\pi_r(x(q)) + wS^\pi_r$ for all $r \in \hat{\mathcal{R}}^\pi$ and all $w \in [w^\pi_{r_{\ell'-1}}, w^\pi_{r_{\ell'}}]$.

The initialization step above established that the hypothesis holds for $\ell = 1$. Next we show how the hypothesis extends to $\ell + 1$.

(1) If $w^\pi_{r_\ell}$ is determined by (30) in Algorithm 1, then $\ell = L$. If $w^\pi_{r_\ell}$ is determined by (24) or (28), and $w^{\min}_\ell > \overline{w}^\pi$, then $\ell = L$. Otherwise, $w^\pi_{r_\ell}$ is determined by (24) or (28), and $w^\pi_{r_\ell} = w^{\min}_\ell \leq \overline{w}^\pi$, or $w^\pi_{r_\ell}$ is determined by (26). In all of these cases, it holds that

$$T^\pi_{r_\ell}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_\ell} \;=\; T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_{\ell+1}} \tag{EC.35}$$

Also, $w^\pi_{r_{\ell+1}}$ is determined by (24) or (26) or (28) or (30) or (31).

Case 1: $w^\pi_{r_{\ell+1}}$ is determined by (24) or (28):

If $w^{\min}_{\ell+1} > \overline{w}^\pi$, then $L = \ell + 1$ and $w^\pi_{r_\ell} \leq w^\pi_{r_{\ell+1}} = \overline{w}^\pi$. Otherwise, $w^\pi_{r_{\ell+1}} = w^{\min}_{\ell+1} \leq \overline{w}^\pi$, $S^\pi_{r_{\ell+1}} > S^\pi_{r_{\ell+2}}$, and $T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_{r_{\ell+1}} = T^\pi_{r_{\ell+2}}(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_{r_{\ell+2}}$. Suppose that $w^\pi_{r_\ell} > w^\pi_{r_{\ell+1}}$. Then

$$\begin{aligned} &\left(S^\pi_{r_{\ell+1}} - S^\pi_{r_{\ell+2}}\right) w^\pi_{r_\ell} \;>\; \left(S^\pi_{r_{\ell+1}} - S^\pi_{r_{\ell+2}}\right) w^\pi_{r_{\ell+1}} \;=\; T^\pi_{r_{\ell+2}}(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) \\ \Rightarrow\quad & T^\pi_{r_\ell}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_\ell} \overset{\text{(EC.35)}}{=} T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_{\ell+1}} \;>\; T^\pi_{r_{\ell+2}}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_{\ell+2}} \end{aligned}$$

which contradicts the induction hypothesis (4). Thus $w^\pi_{r_\ell} \leq w^\pi_{r_{\ell+1}} \leq \overline{w}^\pi$.

Case 2: $w^\pi_{r_{\ell+1}}$ is determined by (26):

Then it follows immediately that $w^\pi_{r_\ell} \leq w^\pi_{r_{\ell+1}} \leq \overline{w}^\pi$.

Case 3: $w^\pi_{r_{\ell+1}}$ is determined by (30) or (31):

Then $w^\pi_{r_{\ell+1}} = \overline{w}^\pi$, and thus $w^\pi_{r_\ell} \leq w^\pi_{r_{\ell+1}} = w^\pi_{r_L} = \overline{w}^\pi$.

(2) Suppose that there is a $r \in \hat{\mathcal{R}}^\pi \setminus \mathcal{R}^\pi_{\min}$ such that $r_\ell < r < r_{\ell+1}$ and $q^\pi_r > 0$. In Algorithm 1, $w^\pi_{r_\ell}$ is determined by (24) or (26) or (28) or (30). If $w^\pi_{r_\ell}$ is determined by (30), then $\ell = L$. The remaining cases are considered next.

Case 1: $w^\pi_{r_\ell}$ is determined by (24), or by (28) and $S^\pi_{r_\ell} > S^\pi_r$.

If $w^{\min}_\ell > \overline{w}^\pi$, then $\ell = L$. Otherwise, $w^\pi_{r_\ell} = w^{\min}_\ell \le \overline{w}^\pi$, and

$$\begin{aligned} w^\pi_{r_\ell} &= w^{\min}_\ell = \frac{T^\pi_{r_{\ell+1}}(x(q)) - T^\pi_{r_\ell}(x(q))}{S^\pi_{r_\ell} - S^\pi_{r_{\ell+1}}} < \frac{T^\pi_r(x(q)) - T^\pi_{r_\ell}(x(q))}{S^\pi_{r_\ell} - S^\pi_r} \\ \Rightarrow \quad T^\pi_{r_\ell}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_\ell} &= T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_{\ell+1}} < T^\pi_r(x(q)) + w^\pi_{r_\ell} S^\pi_r \end{aligned}$$

It follows from the induction hypothesis that $w^\pi_{r_\ell} \in (G^\pi)^{-1}\left(p^\pi_{r_\ell}\right)$. Also, note that if $q^\pi_r > 0$ then $p^\pi_{r_\ell} < p^\pi_r \le p^\pi_{r'}$ for all $r' \ge r$. Thus $w^\pi_{r_\ell} \le \hat{w}^\pi_{r'}$ for all $\hat{w}^\pi_{r'} \in (G^\pi)^{-1}\left(p^\pi_{r'}\right)$, and all $r' \ge r$. Consider the directional derivative of Υ at q in the feasible direction $e^\pi_{r_{\ell+1}} - e^\pi_r$:

$$\begin{aligned} &\max\left\{\left(D^\pi\left[T^\pi_{r_{\ell+1}}(x(q)) + \sum_{r'=r_{\ell+1}}^{\hat{R}^\pi-1}\left(S^\pi_{r'} - S^\pi_{r'+1}\right)\hat{w}^\pi_{r'}\right] + \lambda^\pi - \mu^\pi_{r_{\ell+1}}\right)\right. \\ &\quad -\left(D^\pi\left[T^\pi_r(x(q)) + \sum_{r'=r}^{\hat{R}^\pi-1}\left(S^\pi_{r'} - S^\pi_{r'+1}\right)\hat{w}^\pi_{r'}\right] + \lambda^\pi\right) : \hat{w}^\pi_1 \in (G^\pi)^{-1}\left(p^\pi_1\right), \ldots, \hat{w}^\pi_{\hat{R}^\pi-1} \in (G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi-1}\right), \\ &\qquad\qquad \left.\mu^\pi_{r_{\ell+1}} \ge 0,\ \mu^\pi_{r_{\ell+1}} q^\pi_{r_{\ell+1}} = 0\right\} \\ &= \max\left\{D^\pi\left[T^\pi_{r_{\ell+1}}(x(q)) - T^\pi_r(x(q)) - \sum_{r'=r}^{r_{\ell+1}-1}\left(S^\pi_{r'} - S^\pi_{r'+1}\right)\hat{w}^\pi_{r'}\right] : \hat{w}^\pi_r \in (G^\pi)^{-1}\left(p^\pi_r\right), \ldots\right. \\ &\qquad\qquad \left.\ldots, \hat{w}^\pi_{r_{\ell+1}-1} \in (G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}-1}\right)\right\} \\ &\le D^\pi\left[T^\pi_{r_{\ell+1}}(x(q)) - T^\pi_r(x(q)) - \sum_{r'=r}^{r_{\ell+1}-1}\left(S^\pi_{r'} - S^\pi_{r'+1}\right)w^\pi_{r_\ell}\right] \\ &= D^\pi\left[T^\pi_{r_{\ell+1}}(x(q)) - T^\pi_r(x(q)) + \left(S^\pi_{r_{\ell+1}} - S^\pi_r\right)w^\pi_{r_\ell}\right] < 0 \end{aligned}$$

which contradicts the optimality of q.

Case 2: $w^\pi_{r_\ell}$ is determined by (26), or by (28) and $S^\pi_{r_\ell} = S^\pi_r$.

Then $T^\pi_{r_\ell}(x(q)) < T^\pi_r(x(q))$ and $S^\pi_{r_\ell} = S^\pi_{r'} = S^\pi_r$ for all $r' = r_\ell, \ldots, r$. Consider the directional derivative of Υ at q in the feasible direction $e^\pi_{r_\ell} - e^\pi_r$:

$$\max\left\{\left(D^\pi\left[T^\pi_{r_\ell}(x(q)) + \sum_{r'=r_\ell}^{\hat{R}^\pi-1}\left(S^\pi_{r'} - S^\pi_{r'+1}\right)\hat{w}^\pi_{r'}\right] + \lambda^\pi - \mu^\pi_{r_\ell}\right)\right.$$

$$-\left(D^\pi\left[T_r^\pi(x(q))+\sum_{r'=r}^{\hat{R}^\pi-1}\left(S^\pi_{r'}-S^\pi_{r'+1}\right)\hat{w}^\pi_{r'}\right]+\lambda^\pi\right):\ \hat{w}^\pi_1\in(G^\pi)^{-1}\left(p^\pi_1\right),\ldots,\hat{w}^\pi_{\hat{R}^\pi-1}\in(G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi-1}\right),$$

$$\mu^\pi_{r_\ell}\geq 0,\ \mu^\pi_{r_\ell}q^\pi_{r_\ell}=0\Bigg\}$$

$$= \max\left\{D^\pi\left[T^\pi_{r_\ell}(x(q))-T^\pi_r(x(q))+\sum_{r'=r_\ell}^{r-1}\left(S^\pi_{r'}-S^\pi_{r'+1}\right)\hat{w}^\pi_{r'}\right]:\ \hat{w}^\pi_{r_\ell}\in(G^\pi)^{-1}\left(p^\pi_{r_\ell}\right),\ldots,\hat{w}^\pi_{r-1}\in(G^\pi)^{-1}\left(p^\pi_{r-1}\right)\right\}$$

$$= D^\pi\left[T^\pi_{r_\ell}(x(q))-T^\pi_r(x(q))\right] < 0$$

which contradicts the optimality of q.

(3) By the induction hypothesis, $w^\pi_{r_\ell}\in(G^\pi)^{-1}\left(p^\pi_{r_\ell}\right)$. By Algorithm 1, $w^\pi_r=w^\pi_{r_\ell}$ for all $r=r_\ell,\ldots,r_{\ell+1}-1$. It was established in (2) above that $q^\pi_r=0$ for all $r\in\{r_\ell+1,\ldots,r_{\ell+1}-1\}$, and thus $p^\pi_r=p^\pi_{r_\ell}$ and $w^\pi_r\in(G^\pi)^{-1}\left(p^\pi_r\right)$ for all $r\in\{r_\ell+1,\ldots,r_{\ell+1}-1\}$. Next we show by contradiction that $w^\pi_{r_{\ell+1}}\in(G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}}\right)$.

Case 1: Suppose that $w^\pi_{r_{\ell+1}}<\underline{w}^\pi_{r_{\ell+1}}$.

Note that $w^\pi_{r_{\ell+1}}<\underline{w}^\pi_{r_{\ell+1}}$ implies that $G^\pi(w^\pi_{r_{\ell+1}})<p^\pi_{r_{\ell+1}}$. It was established in (1) above that $w^\pi_{r_\ell}=\cdots=w^\pi_{r_{\ell+1}-1}\leq w^\pi_{r_{\ell+1}}$. It follows from $\underline{w}^\pi_{r_{\ell+1}-1}\leq w^\pi_{r_{\ell+1}-1}\leq w^\pi_{r_{\ell+1}}<\underline{w}^\pi_{r_{\ell+1}}$ that $p^\pi_{r_{\ell+1}-1}<p^\pi_{r_{\ell+1}}$, that is, $q^\pi_{r_{\ell+1}}>0$. In Algorithm 1, $w^\pi_{r_{\ell+1}}$ is determined by (24) or (26) or (28) or (30) or (31). It follows from $w^\pi_{r_{\ell+1}}<\underline{w}^\pi_{r_{\ell+1}}$ that $w^\pi_{r_{\ell+1}}$ is not determined by (26). Also, it follows from $w^\pi_{r_{\ell+1}}<\underline{w}^\pi_{r_{\ell+1}}\leq\overline{w}^\pi$ that $w^\pi_{r_{\ell+1}}$ is not determined by (30) or (31). Thus $w^\pi_{r_{\ell+1}}$ is determined by (24) or (28). In both cases, $S^\pi_{r_{\ell+1}}>S^\pi_{r_{\ell+2}}$, and $w^\pi_r=w^\pi_{r_{\ell+1}}$ for all $r\in\{r_{\ell+1},\ldots,r_{\ell+2}-1\}$. It follows from $w^\pi_{r_{\ell+1}}<\underline{w}^\pi_{r_{\ell+1}}\leq\overline{w}^\pi$ that

$$w^\pi_{r_{\ell+1}} = w^{\min}_{\ell+1} = \min\left\{\frac{T^\pi_r(x(q))-T^\pi_{r_{\ell+1}}(x(q))}{S^\pi_{r_{\ell+1}}-S^\pi_r}:\ S^\pi_{r_{\ell+1}}>S^\pi_r,\ r\in\hat{\mathcal{R}}^\pi\right\} = \frac{T^\pi_{r_{\ell+2}}(x(q))-T^\pi_{r_{\ell+1}}(x(q))}{S^\pi_{r_{\ell+1}}-S^\pi_{r_{\ell+2}}}$$

$$\Rightarrow\quad T^\pi_{r_{\ell+1}}(x(q))+w^\pi_{r_{\ell+1}}S^\pi_{r_{\ell+1}} = T^\pi_{r_{\ell+2}}(x(q))+w^\pi_{r_{\ell+1}}S^\pi_{r_{\ell+2}}$$

$$\Rightarrow\quad T^\pi_{r_{\ell+1}}(x(q))+wS^\pi_{r_{\ell+1}} > T^\pi_{r_{\ell+2}}(x(q))+wS^\pi_{r_{\ell+2}}\quad\forall\ w>w^\pi_{r_{\ell+1}} \tag{EC.36}$$

It follows from $q^\pi_{r_{\ell+1}}>0$ that direction $e^\pi_{r_{\ell+2}}-e^\pi_{r_{\ell+1}}$ is feasible. Next, consider the directional derivative of Υ at q in the direction $e^\pi_{r_{\ell+2}}-e^\pi_{r_{\ell+1}}$:

$$\max\left\{\left(D^\pi\left[T^\pi_{r_{\ell+2}}(x(q))+\sum_{r=r_{\ell+2}}^{\hat{R}^\pi-1}\left(S^\pi_r-S^\pi_{r+1}\right)\hat{w}^\pi_r\right]+\lambda^\pi-\mu^\pi_{r_{\ell+2}}\right)\right.$$

$$\left.-\left(D^\pi\left[T^\pi_{r_{\ell+1}}(x(q))+\sum_{r=r_{\ell+1}}^{\hat{R}^\pi-1}\left(S^\pi_r-S^\pi_{r+1}\right)\hat{w}^\pi_r\right]+\lambda^\pi\right):\ \hat{w}^\pi_1\in(G^\pi)^{-1}\left(p^\pi_1\right),\ldots,\hat{w}^\pi_{\hat{R}^\pi-1}\in(G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi-1}\right),\right.$$

$$\mu^\pi_{r_{\ell+2}} \geq 0,\ \mu^\pi_{r_{\ell+2}} q^\pi_{r_{\ell+2}} = 0 \Bigg\}$$

$$= \max \Bigg\{ D^\pi \left[T^\pi_{r_{\ell+2}}(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) - \sum_{r=r_{\ell+1}}^{r_{\ell+2}-1} \left(S^\pi_r - S^\pi_{r+1}\right) \hat{w}^\pi_r \right] : \hat{w}^\pi_{r_{\ell+1}} \in (G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}}\right), \ldots$$

$$\ldots, \hat{w}^\pi_{r_{\ell+2}-1} \in (G^\pi)^{-1}\left(p^\pi_{r_{\ell+2}-1}\right) \Bigg\}$$

$$\leq D^\pi \left[T^\pi_{r_{\ell+2}}(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) - \sum_{r=r_{\ell+1}}^{r_{\ell+2}-1} \left(S^\pi_r - S^\pi_{r+1}\right) \underline{w}^\pi_{r_{\ell+1}} \right]$$

$$= D^\pi \left[T^\pi_{r_{\ell+2}}(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) + \left(S^\pi_{r_{\ell+2}} - S^\pi_{r_{\ell+1}}\right) \underline{w}^\pi_{r_{\ell+1}} \right] \overset{\text{(EC.36)}}{<} 0$$

which contradicts the optimality of q.

Case 2: Suppose that $w^\pi_{r_{\ell+1}} > \overline{w}^\pi_{r_{\ell+1}}$.

Note that $w^\pi_{r_{\ell+1}} > \overline{w}^\pi_{r_{\ell+1}}$ implies that $1 \geq G^\pi(w^\pi_{r_{\ell+1}}) > p^\pi_{r_{\ell+1}}$. It follows from $p^\pi_{r_{\ell+1}} < 1$ that $\left\{ r \in \{r_{\ell+1}+1, \ldots, \hat{R}^\pi\} : q^\pi_r > 0 \right\} \neq \emptyset$. Let $\check{r} := \min\left\{ r \in \{r_{\ell+1}+1, \ldots, \hat{R}^\pi\} : q^\pi_r > 0 \right\}$. It follows that $p^\pi_r = p^\pi_{r_{\ell+1}}$, and $(G^\pi)^{-1}\left(p^\pi_r\right) = (G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}}\right) = [\underline{w}^\pi_{r_{\ell+1}}, \overline{w}^\pi_{r_{\ell+1}}]$ for all $r \in \{r_{\ell+1}, \ldots, \check{r}-1\}$.

Case 2.1: $S^\pi_{r_{\ell+1}} = S^\pi_{\check{r}}$.

Then $w^\pi_{r_{\ell+1}}$ is determined by (26) or (28) or (30). It follows from $w^\pi_{r_{\ell+1}-1} \leq \overline{w}^\pi_{r_{\ell+1}-1} \leq \overline{w}^\pi_{r_{\ell+1}} < w^\pi_{r_{\ell+1}}$ that $w^\pi_{r_{\ell+1}}$ is not determined by (26). If $w^\pi_{r_{\ell+1}}$ is determined by (28) or (30), then $T^\pi_{\check{r}}(x(q)) \geq T^{\min}_{\ell+1} > T^\pi_{r_{\ell+1}}(x(q))$, and thus $T^\pi_{\check{r}}(x(q)) + wS^\pi_{\check{r}} > T^\pi_{r_{\ell+1}}(x(q)) + wS^\pi_{r_{\ell+1}}$ for all w, which contradicts the optimality of q with $q^\pi_{\check{r}} > 0$.

Case 2.2: $S^\pi_{r_{\ell+1}} > S^\pi_{\check{r}}$.

Then $w^\pi_{r_{\ell+1}}$ is determined by (24) or (26) or (28). It follows from $w^\pi_{r_{\ell+1}-1} \leq \overline{w}^\pi_{r_{\ell+1}-1} \leq \overline{w}^\pi_{r_{\ell+1}} < w^\pi_{r_{\ell+1}}$ that $w^\pi_{r_{\ell+1}}$ is not determined by (26). If $w^\pi_{r_{\ell+1}}$ is determined by (24) or (28), then

$$w^\pi_{r_{\ell+1}} \leq w^{\min}_{\ell+1} = \min\left\{ \frac{T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q))}{S^\pi_{r_{\ell+1}} - S^\pi_r} : S^\pi_{r_{\ell+1}} > S^\pi_r,\ r \in \hat{\mathcal{R}}^\pi \right\} \leq \frac{T^\pi_{\check{r}}(x(q)) - T^\pi_{r_{\ell+1}}(x(q))}{S^\pi_{r_{\ell+1}} - S^\pi_{\check{r}}}$$

$$\Rightarrow \quad T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_{r_{\ell+1}} \leq T^\pi_{\check{r}}(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_{\check{r}}$$

$$\Rightarrow \quad T^\pi_{r_{\ell+1}}(x(q)) + wS^\pi_{r_{\ell+1}} < T^\pi_{\check{r}}(x(q)) + wS^\pi_{\check{r}} \quad \forall\ w < w^\pi_{r_{\ell+1}} \tag{EC.37}$$

It follows from $q^\pi_{\check{r}} > 0$ that direction $e^\pi_{r_{\ell+1}} - e^\pi_{\check{r}}$ is feasible. Next, consider the directional derivative of Υ at q in the direction $e^\pi_{r_{\ell+1}} - e^\pi_{\check{r}}$:

$$\max \left\{ \left(D^\pi \left[T^\pi_{r_{\ell+1}}(x(q)) + \sum_{r=r_{\ell+1}}^{\hat{R}^\pi-1} \left(S^\pi_r - S^\pi_{r+1}\right) \hat{w}^\pi_r \right] + \lambda^\pi - \mu^\pi_{r_{\ell+1}} \right) \right.$$

$$-\left(D^\pi\left[T^\pi_{\check{r}}(x(q))+\sum_{r=\check{r}}^{\hat{R}^\pi-1}\left(S^\pi_r-S^\pi_{r+1}\right)\hat{w}^\pi_r\right]+\lambda^\pi\right):\ \hat{w}^\pi_1\in(G^\pi)^{-1}\left(p^\pi_1\right),\ldots,\hat{w}^\pi_{\hat{R}^\pi-1}\in(G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi-1}\right),$$
$$\mu^\pi_{r_{\ell+1}}\geq 0,\ \mu^\pi_{r_{\ell+1}}q^\pi_{r_{\ell+1}}=0\Bigg\}$$

$$\begin{aligned}
&= \max\left\{D^\pi\left[T^\pi_{r_{\ell+1}}(x(q))-T^\pi_{\check{r}}(x(q))+\sum_{r=r_{\ell+1}}^{\check{r}-1}\left(S^\pi_r-S^\pi_{r+1}\right)\hat{w}^\pi_r\right]:\ \hat{w}^\pi_{r_{\ell+1}}\in(G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}}\right),\ldots\right.\\
&\qquad\left.\ldots,\hat{w}^\pi_{\check{r}-1}\in(G^\pi)^{-1}\left(p^\pi_{\check{r}-1}\right)\right\}\\
&= D^\pi\left[T^\pi_{r_{\ell+1}}(x(q))-T^\pi_{\check{r}}(x(q))+\sum_{r=r_{\ell+1}}^{\check{r}-1}\left(S^\pi_r-S^\pi_{r+1}\right)\overline{w}^\pi_{r_{\ell+1}}\right]\\
&= D^\pi\left[T^\pi_{r_{\ell+1}}(x(q))-T^\pi_{\check{r}}(x(q))+\left(S^\pi_{r_{\ell+1}}-S^\pi_{\check{r}}\right)\overline{w}^\pi_{r_{\ell+1}}\right]\ \overset{\text{(EC.37)}}{<}\ 0
\end{aligned}$$

which contradicts the optimality of q.

(4) We show that the following inequalities hold, which implies the induction hypothesis for $\ell+1$:

$$T^\pi_{r_{\ell+1}}(x(q))+wS^\pi_{r_{\ell+1}}\ \leq\ T^\pi_r(x(q))+wS^\pi_r\qquad\forall\, r\in\hat{\mathcal{R}}^\pi\text{ s.t. }S^\pi_r\geq S^\pi_{r_{\ell+1}},\ \forall\, w\geq w^\pi_{r_\ell}\quad\text{(EC.38)}$$

$$T^\pi_{r_{\ell+1}}(x(q))+wS^\pi_{r_{\ell+1}}\ \leq\ T^\pi_r(x(q))+wS^\pi_r\qquad\forall\, r\in\hat{\mathcal{R}}^\pi\text{ s.t. }S^\pi_r<S^\pi_{r_{\ell+1}},\ \forall\, w\leq w^\pi_{r_{\ell+1}}\quad\text{(EC.39)}$$

In Algorithm 1, if $w^\pi_{r_\ell}$ is determined by (24) or (28), and $w^{\min}_\ell>\overline{w}^\pi$, or $w^\pi_{r_\ell}$ is determined by (30), then $\ell=L$. Otherwise, $w^\pi_{r_\ell}$ is determined by (24) or (28), and $w^{\min}_\ell\leq\overline{w}^\pi$, or $w^\pi_{r_\ell}$ is determined by (26). In all these cases, $w^\pi_{r_\ell}=w^{\min}_\ell\leq\overline{w}^\pi$ and

$$T^\pi_{r_\ell}(x(q))+w^\pi_{r_\ell}S^\pi_{r_\ell}\ =\ T^\pi_{r_{\ell+1}}(x(q))+w^\pi_{r_\ell}S^\pi_{r_{\ell+1}}$$

It follows from the hypothesis that

$$T^\pi_{r_\ell}(x(q))+w^\pi_{r_\ell}S^\pi_{r_\ell}\ \leq\ T^\pi_r(x(q))+w^\pi_{r_\ell}S^\pi_r\qquad\forall\, r\in\hat{\mathcal{R}}^\pi$$

Thus

$$\begin{aligned}
&T^\pi_{r_{\ell+1}}(x(q))+w^\pi_{r_\ell}S^\pi_{r_{\ell+1}}\ \leq\ T^\pi_r(x(q))+w^\pi_{r_\ell}S^\pi_r\qquad\forall\, r\in\hat{\mathcal{R}}^\pi\\
\Rightarrow\quad&T^\pi_{r_{\ell+1}}(x(q))+wS^\pi_{r_{\ell+1}}\ \leq\ T^\pi_r(x(q))+wS^\pi_r\qquad\forall\, r\in\hat{\mathcal{R}}^\pi\text{ s.t. }S^\pi_r\geq S^\pi_{r_{\ell+1}},\ \forall\, w\geq w^\pi_{r_\ell}
\end{aligned}$$

Thus, (EC.38) holds for $\ell+1$. Next we show that (EC.39) holds for $\ell+1$.

Case 1: $w^\pi_{r_{\ell+1}}$ is determined by (24) or (28).

Then

$$\begin{aligned} & w^\pi_{r_{\ell+1}} \;\le\; w^{\min}_{\ell+1} \;=\; \min\left\{\frac{T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q))}{S^\pi_{r_{\ell+1}} - S^\pi_r} \;:\; S^\pi_{r_{\ell+1}} > S^\pi_r,\; r \in \hat{\mathcal{R}}^\pi\right\} \\ \Rightarrow\quad & T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_{r_{\ell+1}} \;\le\; T^\pi_r(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi \text{ s.t. } S^\pi_{r_{\ell+1}} > S^\pi_r \\ \Rightarrow\quad & T^\pi_{r_{\ell+1}}(x(q)) + w S^\pi_{r_{\ell+1}} \;\le\; T^\pi_r(x(q)) + w S^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi \text{ s.t. } S^\pi_r < S^\pi_{r_{\ell+1}},\; \forall\, w \le w^\pi_{r_{\ell+1}} \end{aligned}$$

and thus (EC.39) holds for $\ell+1$.

Case 2: $w^\pi_{r_{\ell+1}}$ is determined by (26).

Case 2.1: $q^\pi_{r_{\ell+1}} = 0$.

It was established in (2) above that $q^\pi_r = 0$ for $r_\ell < r < r_{\ell+1}$. Thus $p^\pi_{r_\ell} = p^\pi_{r_{\ell+1}}$. It follows from the hypothesis that $w^\pi_{r_\ell} \in (G^\pi)^{-1}\left(p^\pi_{r_\ell}\right)$, and thus

$$w^\pi_{r_{\ell+1}} \;=\; \max\left\{w^\pi_{r_\ell},\, \min\left\{w \in (G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}}\right)\right\}\right\} \;=\; \max\left\{w^\pi_{r_\ell},\, \min\left\{w \in (G^\pi)^{-1}\left(p^\pi_{r_\ell}\right)\right\}\right\} \;=\; w^\pi_{r_\ell}$$

It also follows from the hypothesis that

$$\begin{aligned} & T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_{\ell+1}} \;=\; T^\pi_{r_\ell}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_\ell} \;\le\; T^\pi_r(x(q)) + w^\pi_{r_\ell} S^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi \\ \Rightarrow\quad & T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_{r_{\ell+1}} \;\le\; T^\pi_r(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi \\ \Rightarrow\quad & T^\pi_{r_{\ell+1}}(x(q)) + w S^\pi_{r_{\ell+1}} \;\le\; T^\pi_r(x(q)) + w S^\pi_r \qquad \forall\, r \in \hat{\mathcal{R}}^\pi \text{ s.t. } S^\pi_r < S^\pi_{r_{\ell+1}},\; \forall\, w \le w^\pi_{r_{\ell+1}} \end{aligned}$$

Thus, (EC.39) holds for $\ell+1$.

Case 2.2: $q^\pi_{r_{\ell+1}} > 0$.

We will show by contradiction that (EC.39) holds for $\ell+1$. First, suppose there is an $\hat{r} \in \hat{\mathcal{R}}^\pi$ such that $S^\pi_{\hat{r}} < S^\pi_{r_{\ell+1}}$ and $T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_{r_{\ell+1}} > T^\pi_{\hat{r}}(x(q)) + w^\pi_{r_{\ell+1}} S^\pi_{\hat{r}}$. Thus $\hat{r} > r_{\ell+1}$. It follows from the hypothesis that $w^\pi_{r_\ell} \in (G^\pi)^{-1}\left(p^\pi_{r_\ell}\right)$. Also, $q^\pi_{r_{\ell+1}} > 0$ implies that $p^\pi_{r_{\ell+1}} > p^\pi_{r_\ell}$, and thus $w^\pi_{r_\ell} \le \min\left\{w \in (G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}}\right)\right\}$. Hence

$$w^\pi_{r_{\ell+1}} \;=\; \max\left\{w^\pi_{r_\ell},\, \min\left\{w \in (G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}}\right)\right\}\right\} \;=\; \min\left\{w \in (G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}}\right)\right\} \;\le\; \min\left\{w \in (G^\pi)^{-1}\left(p^\pi_r\right)\right\} \tag{EC.40}$$

for all $r \in \hat{\mathcal{R}}^\pi$ such that $r \ge r_{\ell+1}$. Then consider the directional derivative of Υ at q in the feasible direction $e^\pi_{\hat{r}} - e^\pi_{r_{\ell+1}}$:

$$\max\left\{\left(D^\pi\left[T^\pi_{\hat{r}}(x(q)) + \sum_{r=\hat{r}}^{\hat{R}^\pi - 1}\left(S^\pi_r - S^\pi_{r+1}\right)\hat{w}^\pi_r\right] + \lambda^\pi - \mu^\pi_{\hat{r}}\right)\right.$$

$$-\left(D^\pi\left[T^\pi_{r_{\ell+1}}(x(q))+\sum_{r=r_{\ell+1}}^{\hat{R}^\pi-1}\left(S^\pi_r-S^\pi_{r+1}\right)\hat{w}^\pi_r\right]+\lambda^\pi\right):\ \hat{w}^\pi_1\in(G^\pi)^{-1}\left(p^\pi_1\right),\ldots,\hat{w}^\pi_{\hat{R}^\pi-1}\in(G^\pi)^{-1}\left(p^\pi_{\hat{R}^\pi-1}\right),$$
$$\mu^\pi_{\hat{r}}\geq 0,\ \mu^\pi_{\hat{r}}q^\pi_{\hat{r}}=0\Bigg\}$$

$$=\ \max\left\{D^\pi\left[T^\pi_{\hat{r}}(x(q))-T^\pi_{r_{\ell+1}}(x(q))-\sum_{r=r_{\ell+1}}^{\hat{r}-1}\left(S^\pi_r-S^\pi_{r+1}\right)\hat{w}^\pi_r\right]:\ \hat{w}^\pi_{r_{\ell+1}}\in(G^\pi)^{-1}\left(p^\pi_{r_{\ell+1}}\right),\ldots\right.$$
$$\left.\ldots,\hat{w}^\pi_{\hat{r}-1}\in(G^\pi)^{-1}\left(p^\pi_{\hat{r}-1}\right)\right\}$$

$$\overset{\text{(EC.40)}}{\leq}\ D^\pi\left[T^\pi_{\hat{r}}(x(q))-T^\pi_{r_{\ell+1}}(x(q))-\sum_{r=r_{\ell+1}}^{\hat{r}-1}\left(S^\pi_r-S^\pi_{r+1}\right)w^\pi_{r_{\ell+1}}\right]$$
$$=\ D^\pi\left[T^\pi_{\hat{r}}(x(q))-T^\pi_{r_{\ell+1}}(x(q))+w^\pi_{r_{\ell+1}}\left(S^\pi_{\hat{r}}-S^\pi_{r_{\ell+1}}\right)\right]\ <\ 0$$

which contradicts the optimality of q. Thus, for all $r\in\hat{\mathcal{R}}^\pi$ such that $S^\pi_r<S^\pi_{r_{\ell+1}}$ it holds that

$$\begin{aligned}T^\pi_{r_{\ell+1}}(x(q))+w^\pi_{r_{\ell+1}}S^\pi_{r_{\ell+1}}\ &\leq\ T^\pi_r(x(q))+w^\pi_{r_{\ell+1}}S^\pi_r\\ \Rightarrow\qquad T^\pi_{r_{\ell+1}}(x(q))+wS^\pi_{r_{\ell+1}}\ &\leq\ T^\pi_r(x(q))+wS^\pi_r\qquad\forall\ w\leq w^\pi_{r_{\ell+1}}\end{aligned}$$

Thus, (EC.39) holds for $\ell+1$.

Case 3: $w^\pi_{r_{\ell+1}}$ is determined by (30).

Then $\left\{r\in\hat{\mathcal{R}}^\pi\ :\ S^\pi_{r_{\ell+1}}>S^\pi_r\right\}=\varnothing$, and thus (EC.39) holds for $\ell+1$.

Thus the hypothesis has been established. It remains to verify that (2) and (3) hold for $r>r_L$.

(2) In Algorithm 1, when $\ell=L$, $w^{\min}_L$ and $w^\pi_{r_L}$ are determined by (24) or (28) or (30), and $w^{\min}_L>\overline{w}^\pi$.

Consider any $r>r_L$ and suppose that $q^\pi_r>0$. Then for any $r'<r$ it holds that $p^\pi_{r'}<1$, and thus any $\hat{w}\in(G^\pi)^{-1}\left(p^\pi_{r'}\right)$ satisfies $\hat{w}\leq\overline{w}^\pi$.

Case 1: $w^\pi_{r_L}$ is determined by (24) or (28) and $S^\pi_{r_L}>S^\pi_r$.

Then

$$\overline{w}^\pi\ <\ w^{\min}_L\ =\ \min\left\{\frac{T^\pi_{r'}(x(q))-T^\pi_{r_L}(x(q))}{S^\pi_{r_L}-S^\pi_{r'}}\ :\ S^\pi_{r_L}>S^\pi_{r'},\ r'\in\hat{\mathcal{R}}^\pi\right\}\ \leq\ \frac{T^\pi_r(x(q))-T^\pi_{r_L}(x(q))}{S^\pi_{r_L}-S^\pi_r}$$
$$\Rightarrow\quad T^\pi_{r_L}(x(q))+\overline{w}^\pi S^\pi_{r_L}\ <\ T^\pi_r(x(q))+\overline{w}^\pi S^\pi_r \tag{EC.41}$$

Next, consider the directional derivative of Υ at q in the feasible direction $e^\pi_{r_L}-e^\pi_r$:

$$\max\left\{\left(D^\pi\left[T^\pi_{r_L}(x(q))+\sum_{r'=r_L}^{\hat{R}^\pi-1}\left(S^\pi_{r'}-S^\pi_{r'+1}\right)\hat{w}^\pi_{r'}\right]+\lambda^\pi-\mu^\pi_{r_L}\right)\right.$$

$$-\left(D^\pi\left[T_r^\pi(x(q))+\sum_{r'=r}^{\hat{R}^\pi-1}\left(S_{r'}^\pi-S_{r'+1}^\pi\right)\hat{w}_{r'}^\pi\right]+\lambda^\pi\right):\ \hat{w}_1^\pi\in(G^\pi)^{-1}\left(p_1^\pi\right),\ldots,\hat{w}_{\hat{R}^\pi-1}^\pi\in(G^\pi)^{-1}\left(p_{\hat{R}^\pi-1}^\pi\right),$$
$$\mu_{r_L}^\pi\geq 0,\ \mu_{r_L}^\pi q_{r_L}^\pi=0\Bigg\}$$

$$= \max\left\{D^\pi\left[T_{r_L}^\pi(x(q))-T_r^\pi(x(q))+\sum_{r'=r_L}^{r-1}\left(S_{r'}^\pi-S_{r'+1}^\pi\right)\hat{w}_{r'}^\pi\right]:\ \hat{w}_{r_L}^\pi\in(G^\pi)^{-1}\left(p_{r_L}^\pi\right),\ldots,\hat{w}_{r-1}^\pi\in(G^\pi)^{-1}\left(p_{r-1}^\pi\right)\right\}$$

$$\leq D^\pi\left[T_{r_L}^\pi(x(q))-T_r^\pi(x(q))+\sum_{r'=r_L}^{r-1}\left(S_{r'}^\pi-S_{r'+1}^\pi\right)\overline{w}^\pi\right]$$

$$= D^\pi\left[T_{r_L}^\pi(x(q))-T_r^\pi(x(q))+\left(S_{r_L}^\pi-S_r^\pi\right)\overline{w}^\pi\right] \overset{\text{(EC.41)}}{<} 0$$

which contradicts the optimality of q.

Case 2: $w_{r_L}^\pi$ is determined by (28) and $S_{r_L}^\pi=S_r^\pi$, or by (30).

Then $T_{r_L}^\pi(x(q))<T_r^\pi(x(q))$ and $S_{r_L}^\pi=S_{r'}^\pi=S_r^\pi$ for all $r'=r_L,\ldots,r$. Next, consider the directional derivative of Υ at q in the feasible direction $e_{r_L}^\pi-e_r^\pi$:

$$\max\left\{\left(D^\pi\left[T_{r_L}^\pi(x(q))+\sum_{r'=r_L}^{\hat{R}^\pi-1}\left(S_{r'}^\pi-S_{r'+1}^\pi\right)\hat{w}_{r'}^\pi\right]+\lambda^\pi-\mu_{r_L}^\pi\right)\right.$$
$$\left.-\left(D^\pi\left[T_r^\pi(x(q))+\sum_{r'=r}^{\hat{R}^\pi-1}\left(S_{r'}^\pi-S_{r'+1}^\pi\right)\hat{w}_{r'}^\pi\right]+\lambda^\pi\right):\ \hat{w}_1^\pi\in(G^\pi)^{-1}\left(p_1^\pi\right),\ldots,\hat{w}_{\hat{R}^\pi-1}^\pi\in(G^\pi)^{-1}\left(p_{\hat{R}^\pi-1}^\pi\right),\right.$$
$$\left.\mu_{r_L}^\pi\geq 0,\ \mu_{r_L}^\pi q_{r_L}^\pi=0\right\}$$

$$= \max\left\{D^\pi\left[T_{r_L}^\pi(x(q))-T_r^\pi(x(q))+\sum_{r'=r_L}^{r-1}\left(S_{r'}^\pi-S_{r'+1}^\pi\right)\hat{w}_{r'}^\pi\right]:\ \hat{w}_{r_L}^\pi\in(G^\pi)^{-1}\left(p_{r_L}^\pi\right),\ldots,\hat{w}_{r-1}^\pi\in(G^\pi)^{-1}\left(p_{r-1}^\pi\right)\right\}$$

$$= D^\pi\left[T_{r_L}^\pi(x(q))-T_r^\pi(x(q))\right] < 0$$

which contradicts the optimality of q. Thus if $r>r_L$ then $q_r^\pi=0$.

(3) It was established by induction that $w_{r_L}^\pi\in(G^\pi)^{-1}\left(p_{r_L}^\pi\right)$. By Algorithm 1, $w_r^\pi=w_{r_L}^\pi=\overline{w}^\pi$ for all $r\geq r_L$. It was established in (2) above that $q_r^\pi=0$ for all $r>r_L$, and thus $p_r^\pi=p_{r_L}^\pi=1$ and $w_r^\pi\in(G^\pi)^{-1}\left(p_r^\pi\right)$ for all $r>r_L$.

□

Lemma 9 *Assume that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous for all $a \in \mathcal{A}$. Consider any optimal solution q for problem* (18)*, and the resulting output w and $\mathcal{R}^\pi_{\min} = \{r_1, \ldots, r_L\}$ of Algorithm 1. Let $\hat{\mathcal{R}}^\pi$ be indexed with $r = 1, \ldots, \hat{R}^\pi$, and let*

$$\lambda^\pi \;:=\; -D^\pi \left[T^\pi_{r_L}(x(q)) + \overline{w}^\pi \left(S^\pi_{r_L} - S^\pi_{\hat{R}^\pi} \right) \right] \qquad \forall\, \pi \in \Pi$$

Then (q, λ, w) satisfies the optimality conditions (21)–(22)*.*

Proof. Consider any $\pi \in \Pi$. Let $p^\pi_r = L^\pi_r(q^\pi)$. Consider any successive pair $r_\ell, r_{\ell+1} \in \mathcal{R}^\pi_{\min}$. If $w^\pi_{r_\ell}$ is determined by (30) in Algorithm 1, then $\ell = L$, and thus this case does not apply here. Similarly, if $w^\pi_{r_\ell}$ is determined by (24) or (28), and $w^{\min}_\ell > \overline{w}^\pi$, then $\ell = L$, and thus these cases also do not apply here. Otherwise, $w^\pi_{r_\ell}$ is determined by (24) or (28), and $w^\pi_{r_\ell} = w^{\min}_\ell \le \overline{w}^\pi$, or $w^\pi_{r_\ell}$ is determined by (26). In all of these cases it holds that

$$T^\pi_{r_\ell}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_\ell} \;=\; T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_{\ell+1}} \tag{EC.42}$$

that is, $w^\pi_{r_\ell}$ is a boundary or indifference point between the interval $[w^\pi_{r_{\ell-1}}, w^\pi_{r_\ell}]$ where r_ℓ has least bicriteria cost and the interval $[w^\pi_{r_\ell}, w^\pi_{r_{\ell+1}}]$ where $r_{\ell+1}$ has least bicriteria cost.

Also, note that $w^\pi_r = w^\pi_{r_\ell}$ for all $r \in \{r_\ell, \ldots, r_{\ell+1} - 1\}$, and thus

$$\sum_{r=r_\ell}^{r_{\ell+1}-1} \left(S^\pi_r - S^\pi_{r+1} \right) w^\pi_r \;=\; \sum_{r=r_\ell}^{r_{\ell+1}-1} \left(S^\pi_r - S^\pi_{r+1} \right) w^\pi_{r_\ell} \;=\; \left(S^\pi_{r_\ell} - S^\pi_{r_{\ell+1}} \right) w^\pi_{r_\ell} \tag{EC.43}$$

Similarly, $w^\pi_r = w^\pi_{r_L} = \overline{w}^\pi$ for all $r \in \{r_L, \ldots, \hat{R}^\pi\}$, and thus

$$\sum_{r=r_L}^{\hat{R}^\pi-1} \left(S^\pi_r - S^\pi_{r+1} \right) w^\pi_r \;=\; \sum_{r=r_L}^{\hat{R}^\pi-1} \left(S^\pi_r - S^\pi_{r+1} \right) \overline{w}^\pi \;=\; \left(S^\pi_{r_L} - S^\pi_{\hat{R}^\pi} \right) \overline{w}^\pi \tag{EC.44}$$

Case 1: $r \in \mathcal{R}^\pi_{\min}$.

Let $r = r_\ell$ for some $\ell \in \{1, \ldots, L\}$. Then

$$D^\pi \left[T^\pi_r(x(q)) + \sum_{\{r' \in \hat{\mathcal{R}}^\pi \,:\, r \preceq r',\, r'+1 \in \hat{\mathcal{R}}^\pi\}} \left(S^\pi_{r'} - S^\pi_{r'+1} \right) w^\pi_{r'} \right] + \lambda^\pi$$

$$\overset{(34),(EC.43),(EC.44)}{=} D^\pi \left[T^\pi_{r_\ell}(x(q)) + \sum_{\ell'=\ell}^{L-1} \left(S^\pi_{r_{\ell'}} - S^\pi_{r_{\ell'+1}} \right) w^\pi_{r_{\ell'}} + \left(S^\pi_{r_L} - S^\pi_{\hat{R}^\pi} \right) \overline{w}^\pi \right] - D^\pi \left[T^\pi_{r_L}(x(q)) + \overline{w}^\pi \left(S^\pi_{r_L} - S^\pi_{\hat{R}^\pi} \right) \right]$$

$$= D^\pi \sum_{\ell'=\ell}^{L-1} \left[T^\pi_{r_{\ell'}}(x(q)) - T^\pi_{r_{\ell'+1}}(x(q)) + \left(S^\pi_{r_{\ell'}} - S^\pi_{r_{\ell'+1}} \right) w^\pi_{r_{\ell'}} \right] \overset{(EC.42)}{=} 0$$

Thus conditions (21)–(22) hold for all $r \in \mathcal{R}^\pi_{\min}$.

Case 2: $r \notin \mathcal{R}^\pi_{\min}$.

It follows from Lemma 8(2) that $q^\pi_r = 0$, and thus condition (21) holds. Next we show that condition (22) also holds.

Case 2.1: $r < r_L$.

Then either $r < r_1$, or there is a successive pair $r_\ell, r_{\ell+1} \in \mathcal{R}^\pi_{\min}$ such that $r_\ell < r < r_{\ell+1}$. If $r < r_1$, then $\ell = 0$ and $w^\pi_{r'} = w^\pi_{r_0} = \underline{w}^\pi$ for all $r' \in \{r, \ldots, r_1 - 1\}$, and if $r_\ell < r < r_{\ell+1}$, then $w^\pi_{r'} = w^\pi_{r_\ell}$ for all $r' \in \{r, \ldots, r_{\ell+1} - 1\}$. Then

$$
\begin{aligned}
& D^\pi \left[T^\pi_r(x(q)) + \sum_{\{r' \in \hat{\mathcal{R}}^\pi : r \preceq r', r'+1 \in \hat{\mathcal{R}}^\pi\}} \left(S^\pi_{r'} - S^\pi_{r'+1}\right) w^\pi_{r'} \right] + \lambda^\pi \\
\overset{\text{(EC.43),(EC.44)}}{=} \; & D^\pi \left[T^\pi_r(x(q)) + \sum_{r'=r}^{r_{\ell+1}-1} \left(S^\pi_{r'} - S^\pi_{r'+1}\right) w^\pi_{r_\ell} + \sum_{\ell'=\ell+1}^{L-1} \left(S^\pi_{r_{\ell'}} - S^\pi_{r_{\ell'+1}}\right) w^\pi_{r_{\ell'}} + \left(S^\pi_{r_L} - S^\pi_{\hat{R}^\pi}\right) \overline{w}^\pi \right] \\
& \quad - D^\pi \left[T^\pi_{r_L}(x(q)) + \overline{w}^\pi \left(S^\pi_{r_L} - S^\pi_{\hat{R}^\pi}\right) \right] \\
= \; & D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) + T^\pi_{r_{\ell+1}}(x(q)) - T^\pi_{r_L}(x(q)) + \left(S^\pi_r - S^\pi_{r_{\ell+1}}\right) w^\pi_{r_\ell} + \sum_{\ell'=\ell+1}^{L-1} \left(S^\pi_{r_{\ell'}} - S^\pi_{r_{\ell'+1}}\right) w^\pi_{r_{\ell'}} \right] \\
= \; & D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) + \left(S^\pi_r - S^\pi_{r_{\ell+1}}\right) w^\pi_{r_\ell} \right] + D^\pi \sum_{\ell'=\ell+1}^{L-1} \left[T^\pi_{r_{\ell'}}(x(q)) - T^\pi_{r_{\ell'+1}}(x(q)) + \left(S^\pi_{r_{\ell'}} - S^\pi_{r_{\ell'+1}}\right) w^\pi_{r_{\ell'}} \right] \\
\overset{\text{(EC.42)}}{=} \; & D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) + \left(S^\pi_r - S^\pi_{r_{\ell+1}}\right) w^\pi_{r_\ell} \right]
\end{aligned}
$$

Case 2.1.0: $r < r_1$.

Then $\ell = 0$ and $w^\pi_{r_\ell} = \underline{w}^\pi$. Thus

$$
T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) + \left(S^\pi_r - S^\pi_{r_{\ell+1}}\right) w^\pi_{r_\ell} = T^\pi_r(x(q)) - T^\pi_{r_1}(x(q)) + \left(S^\pi_r - S^\pi_{r_1}\right) \underline{w}^\pi \overset{(23)}{\geq} 0
$$

Case 2.1.1: $r_1 < r < r_L$.

Then there is a successive pair $r_\ell, r_{\ell+1} \in \mathcal{R}^\pi_{\min}$ such that $r_\ell < r < r_{\ell+1}$.

Case 2.1.1.1: $w^\pi_{r_\ell}$ is determined by (24) or (28) and $S^\pi_{r_\ell} > S^\pi_r$.

Then

$$
\begin{aligned}
& w^\pi_{r_\ell} = w^{\min}_\ell = \frac{T^\pi_{r_{\ell+1}}(x(q)) - T^\pi_{r_\ell}(x(q))}{S^\pi_{r_\ell} - S^\pi_{r_{\ell+1}}} < \frac{T^\pi_r(x(q)) - T^\pi_{r_\ell}(x(q))}{S^\pi_{r_\ell} - S^\pi_r} \\
\Rightarrow \quad & T^\pi_{r_{\ell+1}}(x(q)) + w^\pi_{r_\ell} S^\pi_{r_{\ell+1}} < T^\pi_r(x(q)) + w^\pi_{r_\ell} S^\pi_r \\
\Rightarrow \quad & D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) + \left(S^\pi_r - S^\pi_{r_{\ell+1}}\right) w^\pi_{r_\ell} \right] > 0
\end{aligned}
$$

(The strict inequality holds because $r_{\ell+1}$ is chosen in (25) and (29) as the *smallest* minimizer.)

Case 2.1.1.2: $w^\pi_{r_\ell}$ is determined by (28) and $S^\pi_{r_\ell} = S^\pi_r$.

Then $T^\pi_{r_\ell}(x(q)) < T^\pi_r(x(q))$. Thus

$$\begin{aligned} & D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) + \left(S^\pi_r - S^\pi_{r_{\ell+1}}\right) w^\pi_{r_\ell} \right] \\ = \; & D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_\ell}(x(q)) + T^\pi_{r_\ell}(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) + \left(S^\pi_{r_\ell} - S^\pi_{r_{\ell+1}}\right) w^\pi_{r_\ell} \right] \\ \overset{\text{(EC.42)}}{=} \; & D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_\ell}(x(q)) \right] \; > \; 0 \end{aligned}$$

Case 2.1.1.3: $w^\pi_{r_\ell}$ is determined by (26).

Then $S^\pi_{r_\ell} = S^\pi_r = S^\pi_{r_{\ell+1}}$ and $T^\pi_{r_\ell}(x(q)) = T^\pi_{r_{\ell+1}}(x(q)) < T^\pi_r(x(q))$, and thus

$$D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) + \left(S^\pi_r - S^\pi_{r_{\ell+1}}\right) w^\pi_{r_\ell} \right] \; = \; D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_{\ell+1}}(x(q)) \right] \; > \; 0$$

Therefore condition (22) holds for all $r < r_L$.

Case 2.2: $r > r_L$.

Then

$$\begin{aligned} & D^\pi \left[T^\pi_r(x(q)) + \sum_{\{r' \in \hat{\mathcal{R}}^\pi \,:\, r \preceq r',\, r'+1 \in \hat{\mathcal{R}}^\pi\}} \left(S^\pi_{r'} - S^\pi_{r'+1}\right) w^\pi_{r'} \right] + \lambda^\pi \\ = \; & D^\pi \left[T^\pi_r(x(q)) + \sum_{r'=r}^{\hat{R}^\pi - 1} \left(S^\pi_{r'} - S^\pi_{r'+1}\right) \overline{w}^\pi \right] - D^\pi \left[T^\pi_{r_L}(x(q)) + \overline{w}^\pi \left(S^\pi_{r_L} - S^\pi_{\hat{R}^\pi} \right) \right] \\ = \; & D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_L}(x(q)) + \left(S^\pi_r - S^\pi_{r_L}\right) \overline{w}^\pi \right] \end{aligned}$$

Case 2.2.1: $w^\pi_{r_L}$ is determined by (24) or (28) and $S^\pi_{r_L} > S^\pi_r$.

Then

$$w^\pi_{r_L} \; = \; \overline{w}^\pi \; < \; w^{\min}_L \; = \; \min \left\{ \frac{T^\pi_{r'}(x(q)) - T^\pi_{r_L}(x(q))}{S^\pi_{r_L} - S^\pi_{r'}} \; : \; S^\pi_{r_L} > S^\pi_{r'} \right\} \; \leq \; \frac{T^\pi_r(x(q)) - T^\pi_{r_L}(x(q))}{S^\pi_{r_L} - S^\pi_r}$$

$$\Rightarrow \qquad T^\pi_{r_L}(x(q)) + \overline{w}^\pi S^\pi_{r_L} \; < \; T^\pi_r(x(q)) + \overline{w}^\pi S^\pi_r$$

$$\Rightarrow \qquad D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_L}(x(q)) + \left(S^\pi_r - S^\pi_{r_L}\right) \overline{w}^\pi \right] \; > \; 0$$

Case 2.2.2: $w^\pi_{r_L}$ is determined by (28) and $S^\pi_{r_L} = S^\pi_r$.

Then $T^\pi_{r_L}(x(q)) < T^\pi_r(x(q))$. Thus

$$D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_L}(x(q)) + \left(S^\pi_r - S^\pi_{r_L}\right) \overline{w}^\pi \right] \; = \; D^\pi \left[T^\pi_r(x(q)) - T^\pi_{r_L}(x(q)) \right] \; > \; 0$$

Case 2.2.3: $w^{\pi}_{r_L}$ is determined by (26).

Then r_{L+1} set in (27) would satisfy $r_{L+1} > r_L$ and $r_{L+1} \in \mathcal{R}^{\pi}_{\min}$, which cannot hold, and thus $w^{\pi}_{r_L}$ is not determined by (26).

Case 2.2.4: $w^{\pi}_{r_L}$ is determined by (30).

Then $S^{\pi}_{r_L} = S^{\pi}_{r} = S^{\pi}_{\hat{R}^{\pi}}$ and $T^{\pi}_{r}(x(q)) > T^{\pi}_{r_L}(x(q))$. Thus

$$D^{\pi}\left[T^{\pi}_{r}(x(q)) - T^{\pi}_{r_L}(x(q)) + \left(S^{\pi}_{r} - S^{\pi}_{r_L}\right)\overline{w}^{\pi}\right] \;=\; D^{\pi}\left[T^{\pi}_{r}(x(q)) - T^{\pi}_{r_L}(x(q))\right] \;>\; 0$$

Therefore conditions (21)–(22) hold for all $r \in \hat{\mathcal{R}}^{\pi}$. □

Lemma 10 *Assume that $t_a : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous for all $a \in \mathcal{A}$. For each $\pi \in \Pi$, let $\mathcal{R}^{\pi}$ denote the complete path set and let $\hat{\mathcal{R}}^{\pi} \subset \mathcal{R}^{\pi}$. Consider any optimal solution $\hat{q}$ for problem* (18) *with path sets $\hat{\mathcal{R}}^{\pi}$, $\pi \in \Pi$. Let $\hat{w} = \left(\hat{w}^{\pi}_{r}, \pi \in \Pi, r \in \{0\} \cup \hat{\mathcal{R}}^{\pi}\right)$ and $\hat{\mathcal{R}}^{\pi}_{\min} = \{\hat{r}_1, \ldots, \hat{r}_{\hat{L}}\}$ denote the output of Algorithm 1 with input $\hat{\mathcal{R}}^{\pi}$ and $\hat{q}$.*

Let $q^* = \left(q^{\pi *}_{r}, \pi \in \Pi, r \in \mathcal{R}^{\pi}\right)$ be as follows: For each $r \in \hat{\mathcal{R}}^{\pi}$ let $q^{\pi *}_{r} = \hat{q}^{\pi}_{r}$, and for each $r \in \mathcal{R}^{\pi} \setminus \hat{\mathcal{R}}^{\pi}$ let $q^{\pi *}_{r} = 0$. Then $x(q^*) = x(\hat{q})$.

Let $w^* = \left(w^{\pi *}_{r}, \pi \in \Pi, r \in \{0\} \cup \mathcal{R}^{\pi}\right)$ and $\mathcal{R}^{\pi *}_{\min} = \{r^*_1, \ldots, r^*_{L^*}\}$ denote the output of Algorithm 1 with input $\mathcal{R}^{\pi}$ and q^*. Let $\mathcal{R}^{\pi}$ be indexed with $r = 1, \ldots, R^{\pi}$, and let

$$\lambda^{\pi *} \;:=\; -D^{\pi}\left[T^{\pi}_{r^*_{L^*}}(x(q^*)) + \overline{w}^{\pi}\left(S^{\pi}_{r^*_{L^*}} - S^{\pi}_{R^{\pi}}\right)\right] \qquad \forall\, \pi \in \Pi$$

For each $\pi \in \Pi$, let $\mathcal{R}^{\pi +}$ denote an optimal set of paths for the parametric shortest path problem (over the complete path set $\mathcal{R}^{\pi}$) with arc costs $t_a(x_a(\hat{q}))$ and s_a. Suppose that $\mathcal{R}^{\pi +} \subset \hat{\mathcal{R}}^{\pi}$ for each $\pi \in \Pi$. Then it holds that (q^*, λ^*, w^*) satisfies the optimality conditions for the complete path set:

$$q^{\pi *}_{r}\left\{D^{\pi}\left[T^{\pi}_{r}(x(q^*)) + \sum_{r'=r}^{R^{\pi}-1}\left(S^{\pi}_{r'} - S^{\pi}_{r'+1}\right)w^{\pi *}_{r'}\right] + \lambda^{\pi *}\right\} \;=\; 0 \qquad \forall\, \pi \in \Pi,\ \forall\, r \in \mathcal{R}^{\pi}$$

$$D^{\pi}\left[T^{\pi}_{r}(x(q^*)) + \sum_{r'=r}^{R^{\pi}-1}\left(S^{\pi}_{r'} - S^{\pi}_{r'+1}\right)w^{\pi *}_{r'}\right] + \lambda^{\pi *} \;\geq\; 0 \qquad \forall\, \pi \in \Pi,\ \forall\, r \in \mathcal{R}^{\pi}$$

Proof. It is easy to check that $x(q^*) = x(\hat{q})$.

Consider any $\pi \in \Pi$. It follows from Lemma 8(4) that for each $\ell \in \{1, \ldots, \hat{L}\}$ it holds that $T^{\pi}_{\hat{r}_\ell}(x(\hat{q})) + wS^{\pi}_{\hat{r}_\ell} \leq T^{\pi}_{r}(x(\hat{q})) + wS^{\pi}_{r}$ for all $r \in \hat{\mathcal{R}}^{\pi} \subset \mathcal{R}^{\pi}$ and all $w \in [\hat{w}^{\pi}_{\hat{r}_{\ell-1}}, \hat{w}^{\pi}_{\hat{r}_\ell}]$. Next we show that this inequality holds for the complete set of paths $r \in \mathcal{R}^{\pi}$. Suppose that for some $\ell \in \{1, \ldots, \hat{L}\}$ and some $w^+ \in [\hat{w}^{\pi}_{\hat{r}_{\ell-1}}, \hat{w}^{\pi}_{\hat{r}_\ell}]$ there is an $r \in \mathcal{R}^{\pi} \setminus \hat{\mathcal{R}}^{\pi}$ such that $T^{\pi}_{r}(x(q^*)) + w^+S^{\pi}_{r} < T^{\pi}_{\hat{r}_\ell}(x(q^*)) + w^+S^{\pi}_{\hat{r}_\ell}$.

Let $r^+ \in \mathcal{R}^{\pi+}$ denote the shortest path produced by the parametric shortest path algorithm when the arc costs are $c_a(w^+, x_a(q^*)) = t_a(x_a(q^*)) + w^+ s_a$. Then $r^+ \in \mathcal{R}^{\pi+} \setminus \hat{\mathcal{R}}^\pi$, which contradicts $\mathcal{R}^{\pi+} \subset \hat{\mathcal{R}}^\pi$. Thus

$$T^\pi_{\hat{r}_\ell}(x(q^*)) + wS^\pi_{\hat{r}_\ell} \;\leq\; T^\pi_r(x(q^*)) + wS^\pi_r \qquad \forall\, r \in \mathcal{R}^\pi,\ \forall\, w \in [\hat{w}^\pi_{\hat{r}_{\ell-1}}, \hat{w}^\pi_{\hat{r}_\ell}] \tag{EC.45}$$

Let $\hat{r}_0 = r^*_0 = 0$, and note that $\hat{w}^\pi_0 = w^{\pi*}_0 = \underline{w}^\pi$. Note that $\mathcal{R}^{\pi+} \subset \hat{\mathcal{R}}^\pi$ does not imply that all efficient paths are in $\hat{\mathcal{R}}^\pi$, because there may be efficient paths $r^*_\ell \in \mathcal{R}^{\pi*}_{\min} \setminus \hat{\mathcal{R}}^\pi$ (and thus $q^{\pi*}_{r^*_\ell} = 0$). Next we show that $\hat{\mathcal{R}}^\pi_{\min} \subset \mathcal{R}^{\pi*}_{\min}$, and that for each $\hat{r}_\ell \in \hat{\mathcal{R}}^\pi_{\min}$, Algorithm 1 produces the same interval $[\hat{w}^\pi_{\hat{r}_{\ell-1}}, \hat{w}^\pi_{\hat{r}_\ell}]$ whether the input is $\hat{\mathcal{R}}^\pi$ and $\hat{q}$ or $\mathcal{R}^\pi$ and q^*. Thereafter the proof can proceed similarly to the proof of Lemma 9. Specifically, next we show by induction on $\ell \in \{1, \ldots, \hat{L}\}$ that

(1) there are indices $\ell^*(1) < \cdots < \ell^*(\hat{L})$ such that for each $\ell \in \{1, \ldots, \hat{L}\}$ it holds that $\hat{r}_\ell = r^*_{\ell^*(\ell)}$, and

(2) for each $\ell \in \{1, \ldots, \hat{L}\}$ it holds that $\hat{w}^\pi_{\hat{r}_{\ell-1}} = w^{\pi*}_{r^*_{\ell^*(\ell)-1}}$.

It follows from $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$, (23), and (EC.45) that

$$T^\pi_{r^*_1}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_1} \;=\; T^\pi_{\hat{r}_1}(x(q^*)) + \underline{w}^\pi S^\pi_{\hat{r}_1} \tag{EC.46}$$

and thus $r^*_1 \leq \hat{r}_1$. If $r^*_1 = \hat{r}_1$, then claim (1) of the hypothesis holds for $\ell = 1$ with $\ell^*(1) = 1$. Also, $\hat{w}^\pi_{\hat{r}_0} = \underline{w}^\pi = w^{\pi*}_{r^*_0}$, and hence claim (2) of the hypothesis holds for $\ell = 1$. Otherwise, $r^*_1 < \hat{r}_1$, and thus $S^\pi_{r^*_1} \geq S^\pi_{\hat{r}_1}$. Let $\check{\ell} := \max\{\ell \in \{1, \ldots, L^*\} : r^*_\ell < \hat{r}_1\}$. Then it follows from Lemma 8(2) that $\hat{q}^\pi_r = 0$ for all $r \in \hat{\mathcal{R}}^\pi$ such that $r < \hat{r}_1$, and thus $q^{\pi*}_r = 0$ for all $r \in \mathcal{R}^\pi$ such that $r < \hat{r}_1$, and $L^\pi_{r^*_\ell}(q^{\pi*}) = 0$ for all $\ell \in \{1, \ldots, \check{\ell}\}$. Also, $S^\pi_{r^*_1} \geq S^\pi_{r^*_{\check{\ell}}} \geq S^\pi_{\hat{r}_1}$.

Case A.1: $S^\pi_{r^*_1} = S^\pi_{r^*_{\check{\ell}}}$.

Then it follows from $r^*_\ell \in \mathcal{R}^{\pi*}_{\min}$ and $S^\pi_{r^*_1} = S^\pi_{r^*_\ell}$ for all $\ell \in \{1, \ldots, \check{\ell}\}$ that $T^\pi_{r^*_1}(x(q^*)) = T^\pi_{r^*_\ell}(x(q^*))$ for all $\ell \in \{1, \ldots, \check{\ell}\}$. Then, in Algorithm 1, $w^{\pi*}_{r^*_\ell}$ is determined by (26) for all $\ell \in \{1, \ldots, \check{\ell} - 1\}$. Thus $w^{\pi*}_{r^*_\ell} = w^{\pi*}_0 = \underline{w}^\pi$ for all $\ell \in \{1, \ldots, \check{\ell} - 1\}$.

Case A.1.1: $S^\pi_{r^*_1} = S^\pi_{r^*_{\check{\ell}}} = S^\pi_{\hat{r}_1}$.

Then it follows from (EC.46) that $T^\pi_{r^*_1}(x(q^*)) = T^\pi_{r^*_{\check{\ell}}}(x(q^*)) = T^\pi_{\hat{r}_1}(x(q^*))$. Then, in Algorithm 1, $w^{\pi*}_{r^*_{\check{\ell}}}$ is determined by (26), $w^{\pi*}_{r^*_{\check{\ell}}} = w^{\pi*}_{r^*_{\check{\ell}-1}} = \underline{w}^\pi$, and $r^*_{\check{\ell}+1} = \hat{r}_1$, hence claim (1) of the hypothesis holds for $\ell = 1$ with $\ell^*(1) = \check{\ell} + 1$. Also, $\hat{w}^\pi_{\hat{r}_0} = \underline{w}^\pi = w^{\pi*}_{r^*_{\check{\ell}}}$, and hence claim (2) of the hypothesis holds for $\ell = 1$.

Case A.1.2: $S^\pi_{r^*_1} = S^\pi_{r^*_{\check{\ell}}} > S^\pi_{\hat{r}_1}$.

Then, in Algorithm 1, $w^{\pi*}_{r^*_{\check{\ell}}}$ is determined by (24) or (28). Then

$$w^{\min}_{\check{\ell}} \;=\; \min\left\{ \frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_r} \;:\; S^\pi_{r^*_{\check{\ell}}} > S^\pi_r \right\}$$

$$\leq \frac{T^\pi_{\hat{r}_1}(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_{\hat{r}_1}} = \frac{T^\pi_{\hat{r}_1}(x(q^*)) - T^\pi_{r^*_1}(x(q^*))}{S^\pi_{r^*_1} - S^\pi_{\hat{r}_1}} \overset{\text{(EC.46)}}{=} \underline{w}^\pi$$

Suppose that $w^{\min}_{\check{\ell}} < \underline{w}^\pi$. Then

$$w^{\min}_{\check{\ell}} = \frac{T^\pi_{r^*_{\check{\ell}+1}}(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_{r^*_{\check{\ell}+1}}} < \underline{w}^\pi$$

$$\Rightarrow \quad T^\pi_{r^*_{\check{\ell}+1}}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_{\check{\ell}+1}} < T^\pi_{r^*_{\check{\ell}}}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_{\check{\ell}}} = T^\pi_{r^*_1}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_1}$$

which contradicts $T^\pi_{r^*_1}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_1} \leq T^\pi_r(x(q^*)) + \underline{w}^\pi S^\pi_r$ for all $r \in \mathcal{R}^\pi$ from (23). Thus $w^{\min}_{\check{\ell}} = \underline{w}^\pi$, and hence

$$w^{\pi*}_{r^*_{\check{\ell}}} = w^{\min}_{\check{\ell}} = \min\left\{ \frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_r} : S^\pi_{r^*_{\check{\ell}}} > S^\pi_r \right\} = \frac{T^\pi_{\hat{r}_1}(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_{\hat{r}_1}}$$

and $r^*_{\check{\ell}+1} = \hat{r}_1$. Hence claim (1) of the hypothesis holds for $\ell = 1$ with $\ell^*(1) = \check{\ell} + 1$. Also, $\hat{w}^\pi_{\hat{r}_0} = \underline{w}^\pi = w^{\min}_{\check{\ell}} = w^{\pi*}_{r^*_{\check{\ell}}}$, and hence claim (2) of the hypothesis holds for $\ell = 1$.

Case A.2: $S^\pi_{r^*_1} > S^\pi_{r^*_{\check{\ell}}}$.

Then $r^*_1 < r^*_{\check{\ell}} < \hat{r}_1$. Let $\tilde{\ell} := \max\left\{ \ell \in \{1, \ldots, \check{\ell} - 1\} : S^\pi_{r^*_1} = S^\pi_{r^*_\ell} \right\}$. Thus, $S^\pi_{r^*_1} = S^\pi_{r^*_\ell}$ and $T^\pi_{r^*_1}(x(q^*)) = T^\pi_{r^*_\ell}(x(q^*))$ for all $\ell \in \{1, \ldots, \tilde{\ell}\}$, and in Algorithm 1, $w^{\pi*}_{r^*_\ell}$ is determined by (26) for all $\ell = 1, \ldots, \tilde{\ell} - 1$, and $\underline{w}^\pi = w^{\pi*}_{r^*_0} = \cdots = w^{\pi*}_{r^*_{\tilde{\ell}-1}}$. Also, in Algorithm 1, $w^{\pi*}_{r^*_{\tilde{\ell}}}$ is determined by (24) or (28). Then

$$w^{\min}_{\tilde{\ell}} = \min\left\{ \frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_{\tilde{\ell}}}(x(q^*))}{S^\pi_{r^*_{\tilde{\ell}}} - S^\pi_r} : S^\pi_{r^*_{\tilde{\ell}}} > S^\pi_r \right\}$$

$$\leq \frac{T^\pi_{\hat{r}_1}(x(q^*)) - T^\pi_{r^*_{\tilde{\ell}}}(x(q^*))}{S^\pi_{r^*_{\tilde{\ell}}} - S^\pi_{\hat{r}_1}} = \frac{T^\pi_{\hat{r}_1}(x(q^*)) - T^\pi_{r^*_1}(x(q^*))}{S^\pi_{r^*_1} - S^\pi_{\hat{r}_1}} \overset{\text{(EC.46)}}{=} \underline{w}^\pi$$

Suppose that $w^{\min}_{\tilde{\ell}} < \underline{w}^\pi$. Then

$$w^{\min}_{\tilde{\ell}} = \frac{T^\pi_{r^*_{\tilde{\ell}+1}}(x(q^*)) - T^\pi_{r^*_{\tilde{\ell}}}(x(q^*))}{S^\pi_{r^*_{\tilde{\ell}}} - S^\pi_{r^*_{\tilde{\ell}+1}}} < \underline{w}^\pi$$

$$\Rightarrow \quad T^\pi_{r^*_{\tilde{\ell}+1}}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_{\tilde{\ell}+1}} < T^\pi_{r^*_{\tilde{\ell}}}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_{\tilde{\ell}}} = T^\pi_{r^*_1}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_1}$$

which contradicts $T^\pi_{r^*_1}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_1} \leq T^\pi_r(x(q^*)) + \underline{w}^\pi S^\pi_r$ for all $r \in \mathcal{R}^\pi$ from (23). Thus $w^{\pi*}_{r^*_{\tilde{\ell}}} = w^{\min}_{\tilde{\ell}} = \underline{w}^\pi$ and $T^\pi_{r^*_{\tilde{\ell}+1}}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_{\tilde{\ell}+1}} = T^\pi_{r^*_{\tilde{\ell}}}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_{\tilde{\ell}}} = T^\pi_{r^*_1}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_1}$. If $\tilde{\ell} + 1 < \check{\ell}$, then

let $\tilde{\tilde{\ell}} := \max\left\{\ell \in \{\tilde{\ell}+1,\ldots,\check{\ell}\} : S^{\pi}_{r^*_{\tilde{\ell}+1}} = S^{\pi}_{r^*_{\ell}}\right\}$. Then the argument is repeated by induction, to obtain that $w^{\pi*}_{r^*_{\check{\ell}-1}} = \underline{w}^{\pi}$ and

$$T^{\pi}_{r^*_{\check{\ell}}}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{r^*_{\check{\ell}}} \;=\; T^{\pi}_{r^*_1}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{r^*_1} \tag{EC.47}$$

Case A.2.1: $S^{\pi}_{r^*_1} > S^{\pi}_{r^*_{\check{\ell}}} = S^{\pi}_{\hat{r}_1}$.
Then

$$\begin{aligned} & T^{\pi}_{\hat{r}_1}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{\hat{r}_1} \overset{\text{(EC.46)}}{=} T^{\pi}_{r^*_1}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{r^*_1} \overset{\text{(EC.47)}}{=} T^{\pi}_{r^*_{\check{\ell}}}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{r^*_{\check{\ell}}} \\ \Rightarrow \quad & T^{\pi}_{\hat{r}_1}(x(q^*)) \;=\; T^{\pi}_{r^*_{\check{\ell}}}(x(q^*)) \end{aligned}$$

Thus, in Algorithm 1, $w^{\pi*}_{r^*_{\check{\ell}}}$ is determined by (26), and $r^*_{\check{\ell}+1} = \hat{r}_1$. Hence claim (1) of the hypothesis holds for $\ell = 1$ with $\ell^*(1) = \check{\ell}+1$. Also, $w^{\pi*}_{r^*_{\check{\ell}}} = \max\left\{w^{\pi*}_{r^*_{\check{\ell}-1}}, \min\left\{w \in (G^{\pi})^{-1}\left(L^{\pi}_{r^*_{\check{\ell}}}(q^{\pi*})\right)\right\}\right\} = \max\left\{w^{\pi*}_{r^*_{\check{\ell}-1}}, \min\left\{w \in (G^{\pi})^{-1}(0)\right\}\right\} = \underline{w}^{\pi}$. Thus $\hat{w}^{\pi}_{\hat{r}_0} = \underline{w}^{\pi} = w^{\pi*}_{r^*_{\check{\ell}}}$, and hence claim (2) of the hypothesis holds for $\ell = 1$.

Case A.2.2: $S^{\pi}_{r^*_1} > S^{\pi}_{r^*_{\check{\ell}}} > S^{\pi}_{\hat{r}_1}$.
Then, in Algorithm 1, $w^{\pi*}_{r^*_{\check{\ell}}}$ is determined by (24) or (28). Then

$$w^{\min}_{\check{\ell}} \;=\; \min\left\{\frac{T^{\pi}_{r}(x(q^*)) - T^{\pi}_{r^*_{\check{\ell}}}(x(q^*))}{S^{\pi}_{r^*_{\check{\ell}}} - S^{\pi}_{r}} : S^{\pi}_{r^*_{\check{\ell}}} > S^{\pi}_{r}\right\} \;\le\; \frac{T^{\pi}_{\hat{r}_1}(x(q^*)) - T^{\pi}_{r^*_{\check{\ell}}}(x(q^*))}{S^{\pi}_{r^*_{\check{\ell}}} - S^{\pi}_{\hat{r}_1}} \overset{\text{(EC.46),(EC.47)}}{=} \underline{w}^{\pi}$$

Suppose that $w^{\min}_{\check{\ell}} < \underline{w}^{(u,v)}$. Then

$$\begin{aligned} & w^{\min}_{\check{\ell}} \;=\; \frac{T^{\pi}_{r^*_{\check{\ell}+1}}(x(q^*)) - T^{\pi}_{r^*_{\check{\ell}}}(x(q^*))}{S^{\pi}_{r^*_{\check{\ell}}} - S^{\pi}_{r^*_{\check{\ell}+1}}} \;<\; \underline{w}^{\pi} \\ \Rightarrow \quad & T^{\pi}_{r^*_{\check{\ell}+1}}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{r^*_{\check{\ell}+1}} \;<\; T^{\pi}_{r^*_{\check{\ell}}}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{r^*_{\check{\ell}}} \overset{\text{(EC.47)}}{=} T^{\pi}_{r^*_1}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{r^*_1} \end{aligned}$$

which contradicts $T^{\pi}_{r^*_1}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{r^*_1} \le T^{\pi}_{r}(x(q^*)) + \underline{w}^{\pi} S^{\pi}_{r}$ for all $r \in \mathcal{R}^{\pi}$ from (23). Thus $w^{\min}_{\check{\ell}} = \underline{w}^{\pi}$, and hence

$$w^{\pi*}_{r^*_{\check{\ell}}} \;=\; w^{\min}_{\check{\ell}} \;=\; \min\left\{\frac{T^{\pi}_{r}(x(q^*)) - T^{\pi}_{r^*_{\check{\ell}}}(x(q^*))}{S^{\pi}_{r^*_{\check{\ell}}} - S^{\pi}_{r}} : S^{\pi}_{r^*_{\check{\ell}}} > S^{\pi}_{r}\right\} \;=\; \frac{T^{\pi}_{\hat{r}_1}(x(q^*)) - T^{\pi}_{r^*_{\check{\ell}}}(x(q^*))}{S^{\pi}_{r^*_{\check{\ell}}} - S^{\pi}_{\hat{r}_1}}$$

and $r^*_{\check{\ell}+1} = \hat{r}_1$. Hence claim (1) of the hypothesis holds for $\ell = 1$ with $\ell^*(1) = \check{\ell}+1$. Also, $\hat{w}^{\pi}_{\hat{r}_0} = \underline{w}^{\pi} = w^{\min}_{\check{\ell}} = w^{\pi*}_{r^*_{\check{\ell}}}$, and hence claim (2) of the hypothesis holds for $\ell = 1$.

Suppose that the hypothesis holds for $\ell \in \{1, \ldots, \hat{L} - 1\}$. Then

$$T^{\pi}_{\hat{r}_\ell}(x(q^*)) + \hat{w}^{\pi}_{\hat{r}_{\ell-1}} S^{\pi}_{\hat{r}_\ell} = T^{\pi}_{r^*_{\ell^*(\ell)}}(x(q^*)) + w^{\pi*}_{r^*_{\ell^*(\ell)-1}} S^{\pi}_{r^*_{\ell^*(\ell)}} \overset{\text{(EC.45)}}{\leq} T^{\pi}_{r}(x(q^*)) + \hat{w}^{\pi}_{\hat{r}_{\ell-1}} S^{\pi}_{r} \qquad \forall\, r \in \mathcal{R}^{\pi} \tag{EC.48}$$

Also, note that $L^{\pi}_{\hat{r}_\ell}(\hat{q}^{\pi}) = L^{\pi}_{r^*_{\ell^*(\ell)}}(q^{\pi*})$. Next we show that $r^*_{\ell^*(\ell)+1} \leq \hat{r}_{\ell+1}$. In Algorithm 1, $\hat{r}_{\ell+1}$ is determined by (25) or (27) or (29).

Case B.1: $\hat{r}_{\ell+1}$ is determined by (25) or (29).

Suppose that $r^*_{\ell^*(\ell)+1} > \hat{r}_{\ell+1}$. Then $S^{\pi}_{r^*_{\ell^*(\ell)+1}} \leq S^{\pi}_{\hat{r}_{\ell+1}} < S^{\pi}_{\hat{r}_\ell} = S^{\pi}_{r^*_{\ell^*(\ell)}}$. Then

$$\begin{aligned} w^{\min}_{\ell^*(\ell)} &= \frac{T^{\pi}_{r^*_{\ell^*(\ell)+1}}(x(q^*)) - T^{\pi}_{r^*_{\ell^*(\ell)}}(x(q^*))}{S^{\pi}_{r^*_{\ell^*(\ell)}} - S^{\pi}_{r^*_{\ell^*(\ell)+1}}} < \frac{T^{\pi}_{\hat{r}_{\ell+1}}(x(q^*)) - T^{\pi}_{r^*_{\ell^*(\ell)}}(x(q^*))}{S^{\pi}_{r^*_{\ell^*(\ell)}} - S^{\pi}_{\hat{r}_{\ell+1}}} \\ &= \frac{T^{\pi}_{\hat{r}_{\ell+1}}(x(\hat{q})) - T^{\pi}_{\hat{r}_\ell}(x(\hat{q}))}{S^{\pi}_{\hat{r}_\ell} - S^{\pi}_{\hat{r}_{\ell+1}}} = \hat{w}^{\pi}_{\hat{r}_\ell} \end{aligned}$$

$$\Rightarrow \quad T^{\pi}_{r^*_{\ell^*(\ell)+1}}(x(q^*)) + \hat{w}^{\pi}_{\hat{r}_\ell} S^{\pi}_{r^*_{\ell^*(\ell)+1}} < T^{\pi}_{r^*_{\ell^*(\ell)}}(x(q^*)) + \hat{w}^{\pi}_{\hat{r}_\ell} S^{\pi}_{r^*_{\ell^*(\ell)}} = T^{\pi}_{\hat{r}_\ell}(x(q^*)) + \hat{w}^{\pi}_{\hat{r}_\ell} S^{\pi}_{\hat{r}_\ell}$$

which contradicts (EC.45).

Case B.2: $\hat{r}_{\ell+1}$ is determined by (27).

Then $S^{\pi}_{\hat{r}_{\ell+1}} = S^{\pi}_{\hat{r}_\ell} = S^{\pi}_{r^*_{\ell^*(\ell)}}$ and $T^{\pi}_{\hat{r}_{\ell+1}}(x(\hat{q})) = T^{\pi}_{\hat{r}_\ell}(x(\hat{q})) = T^{\pi}_{r^*_{\ell^*(\ell)}}(x(q^*))$. Then $\left\{ r \in \{r^*_{\ell^*(\ell)} + 1, \ldots, R^{\pi}\} : S^{\pi}_{r} = S^{\pi}_{r^*_{\ell^*(\ell)}} \right\} \neq \emptyset$. Suppose that there is an $\check{r} \in \{r^*_{\ell^*(\ell)} + 1, \ldots, R^{\pi}\}$ such that $S^{\pi}_{\check{r}} = S^{\pi}_{r^*_{\ell^*(\ell)}}$ and $T^{\pi}_{\check{r}}(x(q^*)) < T^{\pi}_{\hat{r}_{\ell+1}}(x(q^*))$. Then $T^{\pi}_{\check{r}}(x(q^*)) + \hat{w}^{\pi}_{\hat{r}_\ell} S^{\pi}_{\check{r}} < T^{\pi}_{\hat{r}_{\ell+1}}(x(q^*)) + \hat{w}^{\pi}_{\hat{r}_\ell} S^{\pi}_{\hat{r}_{\ell+1}}$ which contradicts (EC.45). Thus it follows from $\hat{\mathcal{R}}^{\pi} \subset \mathcal{R}^{\pi}$ that $\min\left\{ T^{\pi}_{r}(x(q^*)) : S^{\pi}_{r} = S^{\pi}_{r^*_{\ell^*(\ell)}},\ r \in \{r^*_{\ell^*(\ell)} + 1, \ldots, R^{\pi}\} \right\} = T^{\pi}_{\hat{r}_{\ell+1}}(x(q^*))$, and $r^*_{\ell^*(\ell)+1} = \min\left\{ \arg\min\left\{ T^{\pi}_{r}(x(q^*)) : S^{\pi}_{r} = S^{\pi}_{r^*_{\ell^*(\ell)}},\ r \in \{r^*_{\ell^*(\ell)} + 1, \ldots, R^{\pi}\} \right\} \right\} \leq \hat{r}_{\ell+1}$. Thus $r^*_{\ell^*(\ell)+1} \leq \hat{r}_{\ell+1}$.

If $r^*_{\ell^*(\ell)+1} = \hat{r}_{\ell+1}$, then claim (1) of the hypothesis holds for $\ell + 1$ with $\ell^*(\ell + 1) = \ell^*(\ell) + 1$. Also, $\hat{w}^{\pi}_{\hat{r}_\ell} = w^{\pi*}_{r^*_{\ell^*(\ell)}} = w^{\pi*}_{r^*_{\ell^*(\ell+1)-1}}$, and hence claim (2) of the hypothesis holds for $\ell + 1$. Otherwise, $r^*_{\ell^*(\ell)+1} < \hat{r}_{\ell+1}$, and thus $S^{\pi}_{r^*_{\ell^*(\ell)+1}} \geq S^{\pi}_{\hat{r}_{\ell+1}}$. Let $\check{\ell} := \max\{\ell' \in \{\ell^*(\ell) + 1, \ldots, L^*\} : r^*_{\ell'} < \hat{r}_{\ell+1}\}$. It follows from Lemma 8(2) that $\hat{q}^{\pi}_{r} = 0$ for all $r \in \hat{\mathcal{R}}^{\pi}$ such that $r \in \{\hat{r}_\ell + 1, \ldots, \hat{r}_{\ell+1} - 1\}$, and thus $L^{\pi}_{r^*_{\ell'}}(q^{\pi*}) = L^{\pi}_{r^*_{\ell^*(\ell)}}(q^{\pi*}) = L^{\pi}_{\hat{r}_\ell}(\hat{q}^{\pi})$ for all $\ell' \in \{\ell^*(\ell), \ldots, \check{\ell}\}$. Also, $S^{\pi}_{\hat{r}_\ell} = S^{\pi}_{r^*_{\ell^*(\ell)}} \geq S^{\pi}_{r^*_{\check{\ell}}} \geq S^{\pi}_{\hat{r}_{\ell+1}}$.

Case C.1: $S^{\pi}_{r^*_{\ell^*(\ell)}} = S^{\pi}_{r^*_{\check{\ell}}}$.

Then it follows from $r^*_{\ell^*(\ell)} \in \mathcal{R}^{\pi*}_{\min}$ and $S^{\pi}_{r^*_{\ell^*(\ell)}} = S^{\pi}_{r^*_{\ell'}}$ for all $\ell' \in \{\ell^*(\ell), \ldots, \check{\ell}\}$ that $T^{\pi}_{r^*_{\ell^*(\ell)}}(x(q^*)) = T^{\pi}_{r^*_{\ell'}}(x(q^*))$ for all $\ell' \in \{\ell^*(\ell), \ldots, \check{\ell}\}$. Then, in Algorithm 1, $w^{\pi*}_{r^*_{\ell'}}$ is determined by (26) for all $\ell' \in$

$\{\ell^*(\ell), \ldots, \check{\ell}-1\}$. Also, it follows from $L^\pi_{r^*_{\ell'}}(q^{\pi*}) = L^\pi_{r^*_{\ell^*(\ell)}}(q^{\pi*}) = L^\pi_{\hat{r}_\ell}(\hat{q}^\pi)$ for all $\ell' \in \{\ell^*(\ell), \ldots, \check{\ell}\}$ and induction on ℓ' that

$$w^{\pi*}_{r^*_{\ell'}} = \max\left\{w^{\pi*}_{r^*_{\ell'}-1}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{r^*_{\ell'}}(q^{\pi*})\right)\right\}\right\} = \max\left\{w^{\pi*}_{r^*_{\ell^*(\ell)-1}}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{r^*_{\ell^*(\ell)}}(q^{\pi*})\right)\right\}\right\}$$
$$= \max\left\{\hat{w}^\pi_{\hat{r}_{\ell-1}}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{\hat{r}_\ell}(\hat{q}^\pi)\right)\right\}\right\} \qquad \forall\, \ell' \in \{\ell^*(\ell), \ldots, \check{\ell}-1\}$$

Case C.1.1: $S^\pi_{r^*_{\ell^*(\ell)}} = S^\pi_{r^*_{\check{\ell}}} = S^\pi_{\hat{r}_{\ell+1}}$.
Then $T^\pi_{r^*_{\ell^*(\ell)}}(x(q^*)) = T^\pi_{r^*_{\check{\ell}}}(x(q^*)) \le T^\pi_{\hat{r}_{\ell+1}}(x(q^*))$. Suppose $T^\pi_{r^*_{\check{\ell}}}(x(q^*)) < T^\pi_{\hat{r}_{\ell+1}}(x(q^*))$. Then $T^\pi_{r^*_{\check{\ell}}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\check{\ell}}} < T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{\hat{r}_{\ell+1}}$, which contradicts (EC.45). Thus $T^\pi_{r^*_{\ell^*(\ell)}}(x(q^*)) = T^\pi_{r^*_{\check{\ell}}}(x(q^*)) = T^\pi_{\hat{r}_{\ell+1}}(x(q^*))$. Then, in Algorithm 1, $w^{\pi*}_{r^*_{\check{\ell}}}$ is determined by (26), and

$$w^{\pi*}_{r^*_{\check{\ell}}} = \max\left\{w^{\pi*}_{r^*_{\check{\ell}}-1}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{r^*_{\check{\ell}}}(q^{\pi*})\right)\right\}\right\} = \max\left\{w^{\pi*}_{r^*_{\ell^*(\ell)-1}}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{r^*_{\ell^*(\ell)}}(q^{\pi*})\right)\right\}\right\}$$
$$= \max\left\{\hat{w}^\pi_{\hat{r}_{\ell-1}}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{\hat{r}_\ell}(\hat{q}^\pi)\right)\right\}\right\} = \hat{w}^\pi_{\hat{r}_\ell}$$

Thus $r^*_{\check{\ell}+1} = \hat{r}_{\ell+1}$, hence claim (1) of the hypothesis holds for $\ell+1$ with $\ell^*(\ell+1) = \check{\ell}+1$. Also, $\hat{w}^\pi_{\hat{r}_\ell} = w^{\pi*}_{r^*_{\check{\ell}}} = w^{\pi*}_{r^*_{\ell^*(\ell+1)-1}}$, and hence claim (2) of the hypothesis holds for $\ell+1$.

Case C.1.2: $S^\pi_{r^*_{\ell^*(\ell)}} = S^\pi_{r^*_{\check{\ell}}} > S^\pi_{\hat{r}_{\ell+1}}$.
Then, in Algorithm 1, $w^{\pi*}_{r^*_{\check{\ell}}}$ is determined by (24) or (28). Then

$$w^{\min}_{\check{\ell}} = \min\left\{\frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_r} \;:\; S^\pi_{r^*_{\check{\ell}}} > S^\pi_r\right\}$$
$$\le \frac{T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_{\hat{r}_{\ell+1}}} = \frac{T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) - T^\pi_{r^*_{\ell^*(\ell)}}(x(q^*))}{S^\pi_{r^*_{\ell^*(\ell)}} - S^\pi_{\hat{r}_{\ell+1}}} = \frac{T^\pi_{\hat{r}_{\ell+1}}(x(\hat{q})) - T^\pi_{\hat{r}_\ell}(x(\hat{q}))}{S^\pi_{\hat{r}_\ell} - S^\pi_{\hat{r}_{\ell+1}}} = \hat{w}^\pi_{\hat{r}_\ell}$$

Suppose that $w^{\min}_{\check{\ell}} < \hat{w}^\pi_{\hat{r}_\ell}$. Then

$$w^{\min}_{\check{\ell}} = \frac{T^\pi_{r^*_{\check{\ell}+1}}(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_{r^*_{\check{\ell}+1}}} = \frac{T^\pi_{r^*_{\check{\ell}+1}}(x(q^*)) - T^\pi_{r^*_{\ell^*(\ell)}}(x(q^*))}{S^\pi_{r^*_{\ell^*(\ell)}} - S^\pi_{r^*_{\check{\ell}+1}}} = \frac{T^\pi_{r^*_{\check{\ell}+1}}(x(q^*)) - T^\pi_{\hat{r}_\ell}(x(q^*))}{S^\pi_{\hat{r}_\ell} - S^\pi_{r^*_{\check{\ell}+1}}}$$
$$< \frac{T^\pi_{\hat{r}_{\ell+1}}(x(\hat{q})) - T^\pi_{\hat{r}_\ell}(x(\hat{q}))}{S^\pi_{\hat{r}_\ell} - S^\pi_{\hat{r}_{\ell+1}}} = \hat{w}^\pi_{\hat{r}_\ell}$$

$$\Rightarrow \qquad T^\pi_{r^*_{\check{\ell}+1}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\check{\ell}+1}} < T^\pi_{\hat{r}_\ell}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{\hat{r}_\ell} = T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{\hat{r}_{\ell+1}}$$

which contradicts (EC.45). Thus $w^{\min}_{\check{\ell}} = \hat{w}^\pi_{\hat{r}_\ell}$, and hence

$$w^{\pi*}_{r^*_{\check{\ell}}} = w^{\min}_{\check{\ell}} = \min\left\{\frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_r} \;:\; S^\pi_{r^*_{\check{\ell}}} > S^\pi_r\right\} = \frac{T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_{\hat{r}_{\ell+1}}}$$

and $r^*_{\check{\ell}+1} = \hat{r}_{\ell+1}$. Hence claim (1) of the hypothesis holds for $\ell+1$ with $\ell^*(\ell+1) = \check{\ell}+1$. Also, $\hat{w}^\pi_{\hat{r}_\ell} = w^{\pi*}_{r^*_{\check{\ell}}} = w^{\pi*}_{r^*_{\ell^*(\ell+1)-1}}$, and hence claim (2) of the hypothesis holds for $\ell+1$.

Case C.2: $S^\pi_{r^*_{\ell^*(\ell)}} > S^\pi_{r^*_{\check{\ell}}}$.

Then $r^*_{\ell^*(\ell)} < r^*_{\check{\ell}} < \hat{r}_{\ell+1}$. Let $\tilde{\ell} := \max\left\{\ell' \in \{\ell^*(\ell), \ldots, \check{\ell}-1\} : S^\pi_{r^*_{\ell^*(\ell)}} = S^\pi_{r^*_{\ell'}}\right\}$. Thus, $S^\pi_{r^*_{\ell^*(\ell)}} = S^\pi_{r^*_{\ell'}}$ and $T^\pi_{r^*_{\ell^*(\ell)}}(x(q^*)) = T^\pi_{r^*_{\ell'}}(x(q^*))$ for all $\ell' \in \{\ell^*(\ell), \ldots, \tilde{\ell}\}$. Then, in Algorithm 1, $w^{\pi*}_{r^*_{\ell'}}$ is determined by (26) for all $\ell' = \ell^*(\ell), \ldots, \tilde{\ell}-1$. Also, it follows from $L^\pi_{r^*_{\ell'}}(q^{\pi*}) = L^\pi_{r^*_{\ell^*(\ell)}}(q^{\pi*}) = L^\pi_{\hat{r}_\ell}(\hat{q}^\pi)$ for all $\ell' \in \{\ell^*(\ell), \ldots, \tilde{\ell}\}$ and induction on ℓ' that

$$\begin{aligned} w^{\pi*}_{r^*_{\ell'}} &= \max\left\{w^{\pi*}_{r^*_{\ell'-1}}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{r^*_{\ell'}}(q^{\pi*})\right)\right\}\right\} = \max\left\{w^{\pi*}_{r^*_{\ell^*(\ell)-1}}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{r^*_{\ell^*(\ell)}}(q^{\pi*})\right)\right\}\right\} \\ &= \max\left\{\hat{w}^\pi_{\hat{r}_{\ell-1}}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{\hat{r}_\ell}(\hat{q}^\pi)\right)\right\}\right\} \qquad \forall\, \ell' \in \{\ell^*(\ell), \ldots, \tilde{\ell}-1\} \end{aligned}$$

Also, in Algorithm 1, $w^{\pi*}_{r^*_{\tilde{\ell}}}$ is determined by (24) or (28). Then

$$\begin{aligned} w^{\min}_{\tilde{\ell}} &= \min\left\{\frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_{\tilde{\ell}}}(x(q^*))}{S^\pi_{r^*_{\tilde{\ell}}} - S^\pi_r} : S^\pi_{r^*_{\tilde{\ell}}} > S^\pi_r\right\} \\ &\le \frac{T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) - T^\pi_{r^*_{\tilde{\ell}}}(x(q^*))}{S^\pi_{r^*_{\tilde{\ell}}} - S^\pi_{\hat{r}_{\ell+1}}} = \frac{T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) - T^\pi_{r^*_{\ell^*(\ell)}}(x(q^*))}{S^\pi_{r^*_{\ell^*(\ell)}} - S^\pi_{\hat{r}_{\ell+1}}} = \frac{T^\pi_{\hat{r}_{\ell+1}}(x(\hat{q})) - T^\pi_{\hat{r}_\ell}(x(\hat{q}))}{S^\pi_{\hat{r}_\ell} - S^\pi_{\hat{r}_{\ell+1}}} = \hat{w}^\pi_{\hat{r}_\ell} \end{aligned}$$

Suppose that $w^{\min}_{\tilde{\ell}} < \hat{w}^\pi_{\hat{r}_\ell}$. Then

$$\begin{aligned} w^{\min}_{\tilde{\ell}} &= \frac{T^\pi_{r^*_{\tilde{\ell}+1}}(x(q^*)) - T^\pi_{r^*_{\tilde{\ell}}}(x(q^*))}{S^\pi_{r^*_{\tilde{\ell}}} - S^\pi_{r^*_{\tilde{\ell}+1}}} < \hat{w}^\pi_{\hat{r}_\ell} \\ \Rightarrow \quad T^\pi_{r^*_{\tilde{\ell}+1}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\tilde{\ell}+1}} &< T^\pi_{r^*_{\tilde{\ell}}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\tilde{\ell}}} = T^\pi_{r^*_{\ell^*(\ell)}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\ell^*(\ell)}} = T^\pi_{\hat{r}_\ell}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{\hat{r}_\ell} \end{aligned}$$

which contradicts (EC.45). Thus $w^{\pi*}_{r^*_{\tilde{\ell}}} = w^{\min}_{\tilde{\ell}} = \hat{w}^\pi_{\hat{r}_\ell}$ and $T^\pi_{r^*_{\tilde{\ell}+1}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\tilde{\ell}+1}} = T^\pi_{r^*_{\tilde{\ell}}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\tilde{\ell}}} = T^\pi_{r^*_{\ell^*(\ell)}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\ell^*(\ell)}} = T^\pi_{\hat{r}_\ell}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{\hat{r}_\ell}$. If $\tilde{\ell}+1 < \check{\ell}$, then let $\tilde{\tilde{\ell}} := \max\left\{\ell' \in \{\tilde{\ell}+1, \ldots, \check{\ell}\} : S^\pi_{r^*_{\tilde{\ell}+1}} = S^\pi_{r^*_{\ell'}}\right\}$. Then the argument is repeated by induction, to obtain that $w^{\pi*}_{r^*_{\check{\ell}-1}} = \hat{w}^\pi_{\hat{r}_\ell}$ and

$$T^\pi_{r^*_{\check{\ell}}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\check{\ell}}} = T^\pi_{r^*_{\ell^*(\ell)}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\ell^*(\ell)}} = T^\pi_{\hat{r}_\ell}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{\hat{r}_\ell} \overset{\text{(EC.45)}}{=} T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{\hat{r}_{\ell+1}} \tag{EC.49}$$

Case C.2.1: $S^\pi_{r^*_{\ell^*(\ell)}} > S^\pi_{r^*_{\check{\ell}}} = S^\pi_{\hat{r}_{\ell+1}}$.

It follows from (EC.49) that $T^\pi_{r^*_{\check{\ell}}}(x(q^*)) = T^\pi_{\hat{r}_{\ell+1}}(x(q^*))$. Thus, in Algorithm 1, $w^{\pi*}_{r^*_{\check{\ell}}}$ is

determined by (26), and $r^*_{\check{\ell}+1} = \hat{r}_{\ell+1}$. Hence claim (1) of the hypothesis holds for $\ell+1$ with $\ell^*(\ell+1) = \check{\ell}+1$. Also, $w^{\pi*}_{r^*_{\check{\ell}}} = \max\left\{w^{\pi*}_{r^*_{\check{\ell}-1}}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{r^*_{\check{\ell}}}(q^{\pi*})\right)\right\}\right\} = \max\left\{w^{\pi*}_{r^*_{\check{\ell}-1}}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{r^*_{\check{\ell}}}(q^{\pi*})\right)\right\}\right\} = \max\left\{\hat{w}^\pi_{\hat{r}_\ell}, \min\left\{w \in (G^\pi)^{-1}\left(L^\pi_{\hat{r}_\ell}(\hat{q}^\pi)\right)\right\}\right\} = \hat{w}^\pi_{\hat{r}_\ell}$, where the last equality follows from Lemma 8(3). Thus $\hat{w}^\pi_{\hat{r}_\ell} = w^{\pi*}_{r^*_{\check{\ell}}} = w^{\pi*}_{r^*_{\ell^*(\ell+1)-1}}$, and hence claim (2) of the hypothesis holds for $\ell+1$.

Case C.2.2: $S^\pi_{r^*_{\ell^*(\ell)}} > S^\pi_{r^*_{\check{\ell}}} > S^\pi_{\hat{r}_{\ell+1}}$.
Then, in Algorithm 1, $w^{\pi*}_{r^*_{\check{\ell}}}$ is determined by (24) or (28). Then

$$w^{\min}_{\check{\ell}} = \min\left\{\frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_r} : S^\pi_{r^*_{\check{\ell}}} > S^\pi_r\right\} \le \frac{T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_{\hat{r}_{\ell+1}}} \overset{\text{(EC.49)}}{=} \hat{w}^\pi_{\hat{r}_\ell}$$

Suppose that $w^{\min}_{\check{\ell}} < \hat{w}^\pi_{\hat{r}_\ell}$. Then

$$w^{\min}_{\check{\ell}} = \frac{T^\pi_{r^*_{\check{\ell}+1}}(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_{r^*_{\check{\ell}+1}}} < \hat{w}^\pi_{\hat{r}_\ell}$$

$$\Rightarrow \quad T^\pi_{r^*_{\check{\ell}+1}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\check{\ell}+1}} < T^\pi_{r^*_{\check{\ell}}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{r^*_{\check{\ell}}} \overset{\text{(EC.49)}}{=} T^\pi_{\hat{r}_\ell}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{\hat{r}_\ell}$$
$$\overset{\text{(EC.45)}}{=} T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) + \hat{w}^\pi_{\hat{r}_\ell} S^\pi_{\hat{r}_{\ell+1}}$$

which contradicts (EC.45). Thus $w^{\min}_{\check{\ell}} = \hat{w}^\pi_{\hat{r}_\ell}$, and hence

$$w^{\pi*}_{r^*_{\check{\ell}}} = w^{\min}_{\check{\ell}} = \min\left\{\frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_r} : S^\pi_{r^*_{\check{\ell}}} > S^\pi_r\right\} = \frac{T^\pi_{\hat{r}_{\ell+1}}(x(q^*)) - T^\pi_{r^*_{\check{\ell}}}(x(q^*))}{S^\pi_{r^*_{\check{\ell}}} - S^\pi_{\hat{r}_{\ell+1}}}$$

and $r^*_{\check{\ell}+1} = \hat{r}_{\ell+1}$. Hence claim (1) of the hypothesis holds for $\ell+1$ with $\ell^*(\ell+1) = \check{\ell}+1$. Also, $\hat{w}^\pi_{\hat{r}_\ell} = w^{\pi*}_{r^*_{\check{\ell}}} = w^{\pi*}_{r^*_{\ell^*(\ell+1)-1}}$, and hence claim (2) of the hypothesis holds for $\ell+1$. Therefore the hypothesis has been established. In particular, it follows that $\hat{\mathcal{R}}^\pi_{\min} \subset \mathcal{R}^{\pi*}_{\min}$.

Next we show that (q^*, λ^*, w^*) satisfies the optimality conditions (36)–(37). Consider any $\pi \in \Pi$ and any successive pair $r^*_\ell, r^*_{\ell+1} \in \mathcal{R}^{\pi*}_{\min}$. If $w^{\pi*}_{r^*_\ell}$ is determined by (24) or (28), then it holds that $S^\pi_{r^*_\ell} > S^\pi_{r^*_{\ell+1}}$ and

$$w^{\pi*}_{r^*_\ell} = \frac{T^\pi_{r^*_{\ell+1}}(x(q^*)) - T^\pi_{r^*_\ell}(x(q^*))}{S^\pi_{r^*_\ell} - S^\pi_{r^*_{\ell+1}}}$$

(Note that if $w^{\min}_\ell > \overline{w}^\pi$ or $w^{\pi*}_{r^*_\ell}$ is determined by (30), then there is no $r^*_{\ell+1} \in \mathcal{R}^{\pi*}_{\min}$.) If $w^{\pi*}_{r^*_\ell}$ is determined by (26), then $S^\pi_{r^*_\ell} = S^\pi_{r^*_{\ell+1}}$ and $T^\pi_{r^*_\ell}(x(q^*)) = T^\pi_{r^*_{\ell+1}}(x(q^*))$, and thus $T^\pi_{r^*_\ell}(x(q^*)) + wS^\pi_{r^*_\ell} = T^\pi_{r^*_{\ell+1}}(x(q^*)) + wS^\pi_{r^*_{\ell+1}}$ for all w. Thus, in all cases it holds that

$$T^\pi_{r^*_\ell}(x(q^*)) + w^{\pi*}_{r^*_\ell} S^\pi_{r^*_\ell} = T^\pi_{r^*_{\ell+1}}(x(q^*)) + w^{\pi*}_{r^*_\ell} S^\pi_{r^*_{\ell+1}} \tag{EC.50}$$

Also, note that $w_r^{\pi*} = w_{r_\ell^*}^{\pi*}$ for all $r \in \{r_\ell^*, \ldots, r_{\ell+1}^* - 1\}$, and thus

$$\sum_{r=r_\ell^*}^{r_{\ell+1}^*-1} \left(S_r^\pi - S_{r+1}^\pi\right) w_r^{\pi*} \;=\; \sum_{r=r_\ell^*}^{r_{\ell+1}^*-1} \left(S_r^\pi - S_{r+1}^\pi\right) w_{r_\ell^*}^{\pi*} \;=\; \left(S_{r_\ell^*}^\pi - S_{r_{\ell+1}^*}^\pi\right) w_{r_\ell^*}^{\pi*} \tag{EC.51}$$

Similarly, $w_r^{\pi*} = w_{r_{L^*}^*}^{\pi*} = \overline{w}^\pi$ for all $r \in \{r_{L^*}^*, \ldots, R^\pi\}$, and thus

$$\sum_{r=r_{L^*}^*}^{R^\pi-1} \left(S_r^\pi - S_{r+1}^\pi\right) w_r^{\pi*} \;=\; \sum_{r=r_{L^*}^*}^{R^\pi-1} \left(S_r^\pi - S_{r+1}^\pi\right) \overline{w}^\pi \;=\; \left(S_{r_{L^*}^*}^\pi - S_{R^\pi}^\pi\right) \overline{w}^\pi \tag{EC.52}$$

Case D.1: $r \in \mathcal{R}_{\min}^{\pi*}$.

Let $r = r_\ell^*$ for some $\ell \in \{1, \ldots, L^*\}$. Then

$$\begin{aligned}
&D^\pi \left[T_r^\pi(x(q^*)) + \sum_{r'=r}^{R^\pi-1} \left(S_{r'}^\pi - S_{r'+1}^\pi\right) w_{r'}^{\pi*} \right] + \lambda^{\pi*} \\
&\overset{(35),(EC.51),(EC.52)}{=} D^\pi \left[T_{r_\ell^*}^\pi(x(q^*)) + \sum_{\ell'=\ell}^{L^*-1} \left(S_{r_{\ell'}^*}^\pi - S_{r_{\ell'+1}^*}^\pi\right) w_{r_{\ell'}^*}^{\pi*} + \left(S_{r_{L^*}^*}^\pi - S_{R^\pi}^\pi\right) \overline{w}^\pi \right] \\
&\qquad\qquad - D^\pi \left[T_{r_{L^*}^*}^\pi(x(q^*)) + \overline{w}^\pi \left(S_{r_{L^*}^*}^\pi - S_{R^\pi}^\pi\right) \right] \\
&= D^\pi \sum_{\ell'=\ell}^{L^*-1} \left[T_{r_{\ell'}^*}^\pi(x(q^*)) - T_{r_{\ell'+1}^*}^\pi(x(q^*)) + \left(S_{r_{\ell'}^*}^\pi - S_{r_{\ell'+1}^*}^\pi\right) w_{r_{\ell'}^*}^{\pi*} \right] \overset{(EC.50)}{=} 0
\end{aligned}$$

Thus conditions (36)–(37) hold for all $r \in \mathcal{R}_{\min}^{\pi*}$.

Case D.2: $r \notin \mathcal{R}_{\min}^{\pi*}$.

It follows from $\hat{\mathcal{R}}_{\min}^\pi \subset \mathcal{R}_{\min}^{\pi*}$ that $r \notin \hat{\mathcal{R}}_{\min}^\pi$. If $r \in \hat{\mathcal{R}}^\pi$, then it follows from Lemma 8(2) that $\hat{q}_r^\pi = 0$, and thus $q_r^{\pi*} = 0$. If $r \in \mathcal{R}^\pi \setminus \hat{\mathcal{R}}^\pi$ then $q_r^{\pi*} = 0$. Thus condition (36) holds. Next we show that condition (37) also holds.

Case D.2.1: $r < r_{L^*}^*$.

Then either $r < r_1^*$, or there is a successive pair $r_\ell^*, r_{\ell+1}^* \in \mathcal{R}_{\min}^{\pi*}$ such that $r_\ell^* < r < r_{\ell+1}^*$. If $r < r_1^*$, then $\ell = 0$ and $w_{r'}^{\pi*} = w_{r_0^*}^{\pi*} = \underline{w}^\pi$ for all $r' \in \{r, \ldots, r_1^* - 1\}$, and if $r_\ell^* < r < r_{\ell+1}^*$, then $w_{r'}^{\pi*} = w_{r_\ell^*}^{\pi*}$ for all $r' \in \{r, \ldots, r_{\ell+1}^* - 1\}$. Then

$$\begin{aligned}
&D^\pi \left[T_r^\pi(x(q^*)) + \sum_{r'=r}^{R^\pi-1} \left(S_{r'}^\pi - S_{r'+1}^\pi\right) w_{r'}^{\pi*} \right] + \lambda^{\pi*} \\
&\overset{(35),(EC.51),(EC.52)}{=} D^\pi \left[T_r^\pi(x(q^*)) + \sum_{r'=r}^{r_{\ell+1}^*-1} \left(S_{r'}^\pi - S_{r'+1}^\pi\right) w_{r_\ell^*}^{\pi*} + \sum_{\ell'=\ell+1}^{L^*-1} \left(S_{r_{\ell'}^*}^\pi - S_{r_{\ell'+1}^*}^\pi\right) w_{r_{\ell'}^*}^{\pi*} + \left(S_{r_{L^*}^*}^\pi - S_{R^\pi}^\pi\right) \overline{w}^\pi \right]
\end{aligned}$$

$$
\begin{aligned}
& - D^\pi \left[T^\pi_{r^*_{L^*}}(x(q^*)) + \overline{w}^\pi \left(S^\pi_{r^*_{L^*}} - S^\pi_{R^\pi} \right) \right] \\
= \; & D^\pi \left[T^\pi_r(x(q^*)) - T^\pi_{r^*_{\ell+1}}(x(q^*)) + T^\pi_{r^*_{\ell+1}}(x(q^*)) - T^\pi_{r^*_{L^*}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{\ell+1}} \right) w^{\pi*}_{r^*_\ell} + \sum_{\ell'=\ell+1}^{L^*-1} \left(S^\pi_{r^*_{\ell'}} - S^\pi_{r^*_{\ell'+1}} \right) w^{\pi*}_{r^*_{\ell'}} \right] \\
= \; & D^\pi \left[T^\pi_r(x(q^*)) - T^\pi_{r^*_{\ell+1}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{\ell+1}} \right) w^{\pi*}_{r^*_\ell} \right] + D^\pi \sum_{\ell'=\ell+1}^{L^*-1} \left[T^\pi_{r^*_{\ell'}}(x(q^*)) - T^\pi_{r^*_{\ell'+1}}(x(q^*)) + \left(S^\pi_{r^*_{\ell'}} - S^\pi_{r^*_{\ell'+1}} \right) w^{\pi*}_{r^*_{\ell'}} \right] \\
\overset{\text{(EC.50)}}{=} \; & D^\pi \left[T^\pi_r(x(q^*)) - T^\pi_{r^*_{\ell+1}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{\ell+1}} \right) w^{\pi*}_{r^*_\ell} \right]
\end{aligned}
$$

If $r < r^*_1$, then $w^{\pi*}_{r^*_\ell} = w^{\pi*}_{r^*_0} = \underline{w}^\pi$ and $r^*_{\ell+1} = r^*_1$. Then it follows from (23) that

$$
T^\pi_r(x(q^*)) - T^\pi_{r^*_{\ell+1}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{\ell+1}} \right) w^{\pi*}_{r^*_\ell} \;=\; \left[T^\pi_r(x(q^*)) + \underline{w}^\pi S^\pi_r \right] - \left[T^\pi_{r^*_1}(x(q^*)) + \underline{w}^\pi S^\pi_{r^*_1} \right] \;>\; 0
$$

Otherwise, if $r > r^*_1$, then consider the following cases.

Case D.2.1.1: $w^{\pi*}_{r^*_\ell}$ is determined by (24) or (28) and $S^\pi_{r^*_\ell} > S^\pi_r$.

Then

$$
\begin{aligned}
& w^{\pi*}_{r^*_\ell} \;=\; w^{\min}_\ell \;=\; \frac{T^\pi_{r^*_{\ell+1}}(x(q^*)) - T^\pi_{r^*_\ell}(x(q^*))}{S^\pi_{r^*_\ell} - S^\pi_{r^*_{\ell+1}}} \;<\; \frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_\ell}(x(q^*))}{S^\pi_{r^*_\ell} - S^\pi_r} \\
\Rightarrow \quad & T^\pi_{r^*_{\ell+1}}(x(q^*)) + w^{\pi*}_{r^*_\ell} S^\pi_{r^*_{\ell+1}} \;<\; T^\pi_r(x(q^*)) + w^{\pi*}_{r^*_\ell} S^\pi_r \\
\Rightarrow \quad & D^\pi \left[T^\pi_r(x(q^*)) - T^\pi_{r^*_{\ell+1}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{\ell+1}} \right) w^{\pi*}_{r^*_\ell} \right] \;>\; 0
\end{aligned}
$$

Case D.2.1.2: $w^{\pi*}_{r^*_\ell}$ is determined by (28) and $S^\pi_{r^*_\ell} = S^\pi_r$.

Then $T^\pi_{r^*_\ell}(x(q^*)) < T^\pi_r(x(q^*))$. Thus

$$
\begin{aligned}
& D^\pi \left[T^\pi_r(x(q^*)) - T^\pi_{r^*_{\ell+1}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{\ell+1}} \right) w^{\pi*}_{r^*_\ell} \right] \\
= \; & D^\pi \left[T^\pi_r(x(q^*)) - T^\pi_{r^*_\ell}(x(q^*)) + T^\pi_{r^*_\ell}(x(q^*)) - T^\pi_{r^*_{\ell+1}}(x(q^*)) + \left(S^\pi_{r^*_\ell} - S^\pi_{r^*_{\ell+1}} \right) w^{\pi*}_{r^*_\ell} \right] \\
\overset{\text{(EC.50)}}{=} \; & D^\pi \left[T^\pi_r(x(q^*)) - T^\pi_{r^*_\ell}(x(q^*)) \right] \;>\; 0
\end{aligned}
$$

Case D.2.1.3: $w^{\pi*}_{r^*_\ell}$ is determined by (26).

Then $S^\pi_{r^*_\ell} = S^\pi_r = S^\pi_{r^*_{\ell+1}}$ and $T^\pi_{r^*_\ell}(x(q^*)) = T^\pi_{r^*_{\ell+1}}(x(q^*)) < T^\pi_r(x(q^*))$, and thus

$$
D^\pi \left[T^\pi_r(x(q^*)) - T^\pi_{r^*_{\ell+1}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{\ell+1}} \right) w^{\pi*}_{r^*_\ell} \right] \;=\; D^\pi \left[T^\pi_r(x(q^*)) - T^\pi_{r^*_{\ell+1}}(x(q^*)) \right] \;>\; 0
$$

Therefore condition (37) holds for all $r < r^*_{L^*}$.

Case D.2.2: $r > r^*_{L^*}$.

Then

$$
D^\pi \left[T^\pi_r(x(q^*)) + \sum_{r'=r}^{R^\pi-1} \left(S^\pi_{r'} - S^\pi_{r'+1} \right) w^{\pi*}_{r'} \right] + \lambda^{\pi*}
$$

$$\overset{(35)}{=} D^\pi\left[T_r^\pi(x(q^*)) + \sum_{r'=r}^{R^\pi-1}\left(S^\pi_{r'} - S^\pi_{r'+1}\right)\overline{w}^\pi\right] - D^\pi\left[T^\pi_{r^*_{L^*}}(x(q^*)) + \overline{w}^\pi\left(S^\pi_{r^*_{L^*}} - S^\pi_{R^\pi}\right)\right]$$
$$= D^\pi\left[T_r^\pi(x(q^*)) - T^\pi_{r^*_{L^*}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{L^*}}\right)\overline{w}^\pi\right]$$

Case D.2.2.1: $w^{\pi*}_{r^*_{L^*}}$ is determined by (24) or (28) and $S^\pi_{r^*_{L^*}} > S^\pi_r$.

Then

$$w^{\pi*}_{r^*_{L^*}} = \overline{w}^\pi < w^{\min}_{L^*} = \min\left\{\frac{T^\pi_{r'}(x(q^*)) - T^\pi_{r^*_{L^*}}(x(q^*))}{S^\pi_{r^*_{L^*}} - S^\pi_{r'}} : S^\pi_{r^*_{L^*}} > S^\pi_{r'}\right\} \le \frac{T^\pi_r(x(q^*)) - T^\pi_{r^*_{L^*}}(x(q^*))}{S^\pi_{r^*_{L^*}} - S^\pi_r}$$

$$\Rightarrow \quad T^\pi_{r^*_{L^*}}(x(q^*)) + \overline{w}^\pi S^\pi_{r^*_{L^*}} < T^\pi_r(x(q^*)) + \overline{w}^\pi S^\pi_r$$

$$\Rightarrow \quad D^\pi\left[T_r^\pi(x(q^*)) - T^\pi_{r^*_{L^*}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{L^*}}\right)\overline{w}^\pi\right] > 0$$

Case D.2.2.2: $w^{\pi*}_{r^*_{L^*}}$ is determined by (28) and $S^\pi_{r^*_{L^*}} = S^\pi_r$.

Then $T^\pi_{r^*_{L^*}}(x(q^*)) < T^\pi_r(x(q^*))$. Thus

$$D^\pi\left[T_r^\pi(x(q^*)) - T^\pi_{r^*_{L^*}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{L^*}}\right)\overline{w}^\pi\right] = D^\pi\left[T_r^\pi(x(q^*)) - T^\pi_{r^*_{L^*}}(x(q^*))\right] > 0$$

Case D.2.2.3: $w^{\pi*}_{r^*_{L^*}}$ is determined by (26).

Then $r^*_{L^*+1}$ set in (27) would satisfy $r^*_{L^*+1} > r^*_{L^*}$ and $r^*_{L^*+1} \in \mathcal{R}^{\pi*}_{\min}$, which cannot hold, and thus $w^\pi_{r^*_{L^*}}$ is not determined by (26).

Case D.2.2.4: $w^{\pi*}_{r^*_{L^*}}$ is determined by (30).

Then $S^\pi_{r^*_{L^*}} = S^\pi_r = S^\pi_{R^\pi}$ and $T^\pi_r(x(q^*)) > T^\pi_{r^*_{L^*}}(x(q^*))$. Thus

$$D^\pi\left[T_r^\pi(x(q^*)) - T^\pi_{r^*_{L^*}}(x(q^*)) + \left(S^\pi_r - S^\pi_{r^*_{L^*}}\right)\overline{w}^\pi\right] = D^\pi\left[T_r^\pi(x(q^*)) - T^\pi_{r^*_{L^*}}(x(q^*))\right] > 0$$

Therefore conditions (36)–(37) hold for all $r \in \mathcal{R}^\pi$. □

EC.4. Supplementary Numerical Data and Results

In this section of the appendix, we provide additional data and numerical results to evaluate the performance of BTA.

Test networks. Table EC.1 gives summary network data for the benchmark networks used in the numerical tests.

Table EC.1 Summary network data for the benchmark networks.

Network	Nodes	Links	Zones	OD records	Total demand	α	β
Braess	4	5	1	2	6	0.15	4
Sioux Falls	24	76	24	576	360,600	0.15	4
Friedrichshain	224	523	23	506	11,205	0.15	4
Berlin Prenzlauerberg	352	749	38	1,406	16,660	0.15	4
Berlin Tiergarten	359	766	26	644	10,755	0.15	4
Berlin Mitte	397	871	36	1,260	11,482	0.15	4
Anaheim	416	914	38	1,406	104,694	0.15	4
Barcelona	930	2,522	97	7,922	184,680	0.15	4
Winnipeg-Asym	948	2,535	135	4,345	1,361,475	0.1	1.5
Berlin M.–P.–F.	974	2,184	98	9,505	23,648	0.15	4
Winnipeg	1,040	2,836	135	4,345	64,784	0.15	4
Terrassa-Asym	1,603	3,264	55	2,215	25,225,747	0.1	1.5
Hessen-Asym	4,660	6,674	195	17,213	71,250,600	0.1	1.5
Goldcoast	4,783	11,140	1,068	1,140,624	139,256	0.15	4
Chicago Regional	12,979	39,018	1,771	3,136,441	1,360,428	0.15	4
Berlin Center	12,981	28,376	865	49,688	168,222	0.15	4
Philadelphia	13,389	40,003	1,489	1,151,166	18,503,872	0.15	4

BTA subroutine parameters. BTA was used with two solvers for solving problem (18) with subsets of paths $\hat{\mathcal{R}}^\pi$ in Line 10 of Algorithm 2. The FISTA method of Sujanani and Monteiro (2024) was used for the smooth instances, and the two-cut adaptive proximal bundle method (Ad-GPB) of Monteiro and Zhang (2024) was used for the nonsmooth instances. Table EC.2 gives the parameter settings chosen for these solvers, including the network-specific initializations for Ad-GPB. FISTA estimates the Lipschitz constant from an initial Lipschitz estimate M_0 with backtracking, using backtracking factor β. We use the unknown-optimum variant of Ad-GPB with a two-cut bundle update; see Monteiro and Zhang (2024) for details.

Numerical results comparing different methods. Table EC.3 compares the computational performance of Algorithm BTA and the methods of Daganzo (1983), Dial (1996), Marcotte et al. (1996), and Xie et al. (2026) on smooth instances i. If method m achieved relative gap $\gamma_{i,m}(t) \le \varepsilon_i = 10^{-5}$ for instance i at some time $t \le T_i$, where $T_i = 3600$ seconds for all instances except Goldcoast, Chicago Regional, and Philadelphia, which used $T_i = 21600$ seconds, then the table shows the solve time $t_{i,m}$ (in seconds), otherwise the table shows the final relative gap $\gamma_{i,m}(T_i)$ achieved by method m in parentheses. Dial saved no feasible solution within the $T_i = 21600$ second cutoff for Chicago Regional and Philadelphia, because it did not complete its initial minimum-path assignment. A summary of these results is given in Table 1 and Figure 6.

Table EC.2 Parameter settings for the two solvers.

Solver	Parameter	Value
FISTA	Restricted-master tolerance $\widehat{\varepsilon}$	10^{-6}
	Backtracking factor β	1.25
	Initial Lipschitz estimate M_0	50
	Metric	fraction (q) coordinates
	Lipschitz estimate	allowed to decrease
	Backtracking acceptance	scaled to iterate magnitude
Ad-GPB	Initial tolerance $\widehat{\varepsilon}$	10^{-3}
	Initial acceptance parameter β_1	1/4
	(τ, λ_1)	$(0.5, 10^{-3})$
	Maximum number of affine cuts in the model	2
Both	Path pool	monotone retention
	Outer stopping rule	PSPP solution adds no new path

Table EC.3 Computational performance on smooth instances i. The table shows either the solve time $t_{i,m}$ (in seconds) for instance i if method m achieved relative gap $\gamma_{i,m}(t) \le \varepsilon_i = 10^{-5}$ at some time $t \le T_i$ (T_i = 6h for Goldcoast, Chicago Regional, and Philadelphia; T_i = 1h otherwise), or otherwise the final relative gap $\gamma_{i,m}(T_i)$ achieved by method m in parentheses, or "—" meaning that the method saved no feasible solution within T_i.

Network	Daganzo	Dial	Marcotte–Zhu	SBA	BTA
Braess	(2.38×10^{-1})	0.00678	0.00784	0.00214	<0.001
Sioux Falls	0.473	0.131	0.0597	0.122	0.032
Friedrichshain	(2.09×10^{-2})	2.86	0.457	0.259	0.045
Berlin Prenzlauerberg	(1.79×10^{-1})	13.7	0.586	2.78	0.168
Berlin Tiergarten	(4.41×10^{-1})	2.13	0.0864	0.143	0.040
Berlin Mitte	(2.00×10^{-2})	4.01	0.214	0.470	0.070
Anaheim	(2.61×10^{-1})	26.9	2.35	1.16	0.321
Barcelona	(24.1)	1750	289	38.3	5.987
Winnipeg-Asym	(6.22×10^{-2})	611	161	113	1.275
Berlin M.-P.-F.	(3.72×10^{-2})	112	1.91	6.47	1.038
Winnipeg	(33.5)	2280	823	88.7	4.673
Terrassa-Asym	(3.59×10^{-1})	1030	593	496	9.935
Hessen-Asym	(3.59×10^{-2})	(1.94×10^{-5})	820	2130	14.657
Goldcoast	(112)	(4.71)	(1.37×10^{-2})	(1.01×10^{-2})	(1.09×10^{-2})
Chicago Regional	(3.28×10^{3})	—	(8.67×10^{-1})	(1.85×10^{4})	(2.03×10^{-1})
Berlin Center	(3.44×10^{4})	(1.29×10^{-2})	(1.00)	(2.60×10^{-3})	25.352
Philadelphia	(2.21)	—	(1.00)	(24.7)	(5.06×10^{-4})

For each smooth instance i, Figures EC.1–EC.3 show the relative gaps $\gamma_{i,m}(t)$ of each of the five methods m as a function of computation time t, with both axes using a logarithmic scale.

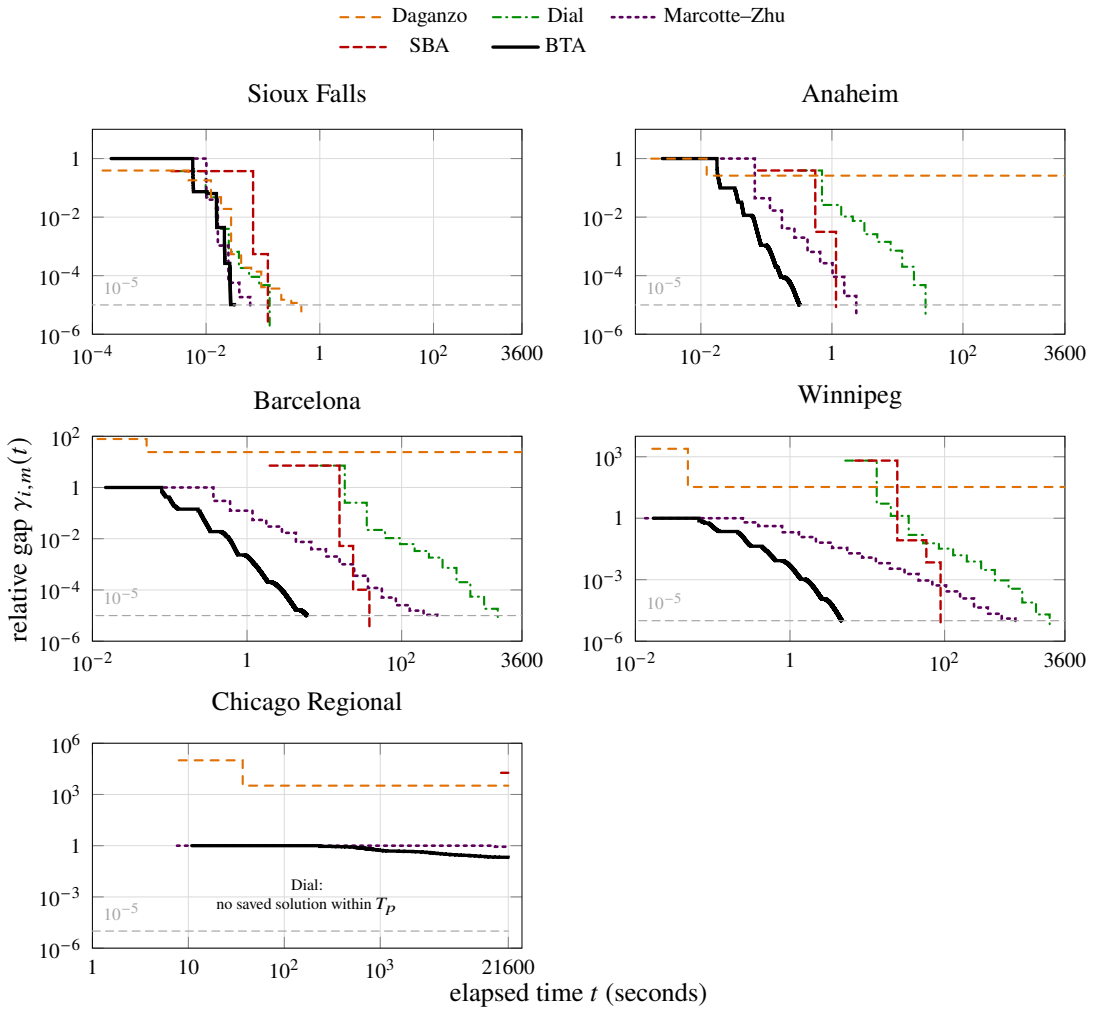


Figure EC.1 For each smooth instances i, the curves show the relative gaps $\gamma_{i,m}(t)$ of the five methods m as a function of computation time t. The dashed line shows the tolerance $\varepsilon_i = 10^{-5}$. Each curve stops at its first recorded point with $\gamma_{i,m}(t) \le \varepsilon_i$, or at T_i.

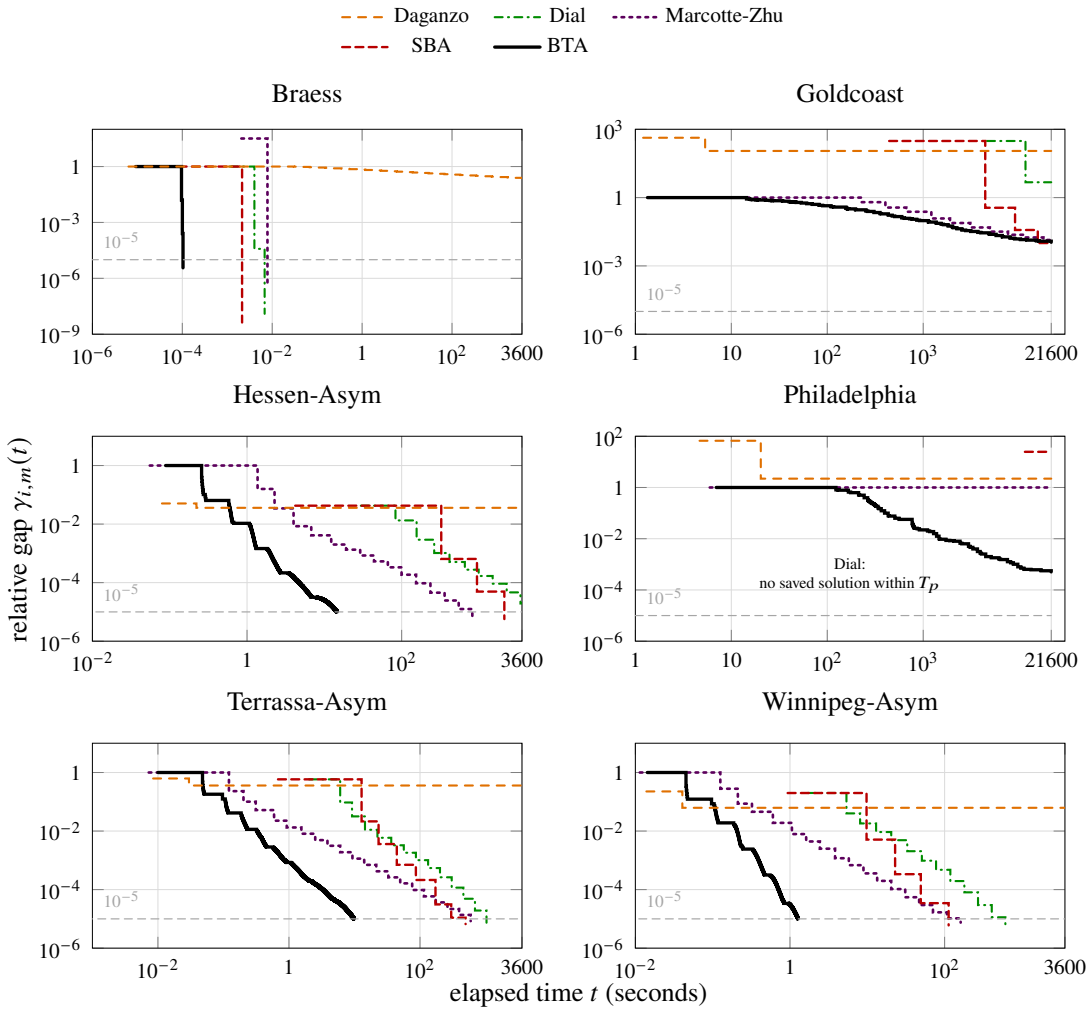


Figure EC.2 For each smooth instances i, the curves show the relative gaps $\gamma_{i,m}(t)$ of the five methods m as a function of computation time t. The dashed line shows the tolerance $\varepsilon_i = 10^{-5}$. Each curve stops at its first recorded point with $\gamma_{i,m}(t) \le \varepsilon_i$, or at T_i.

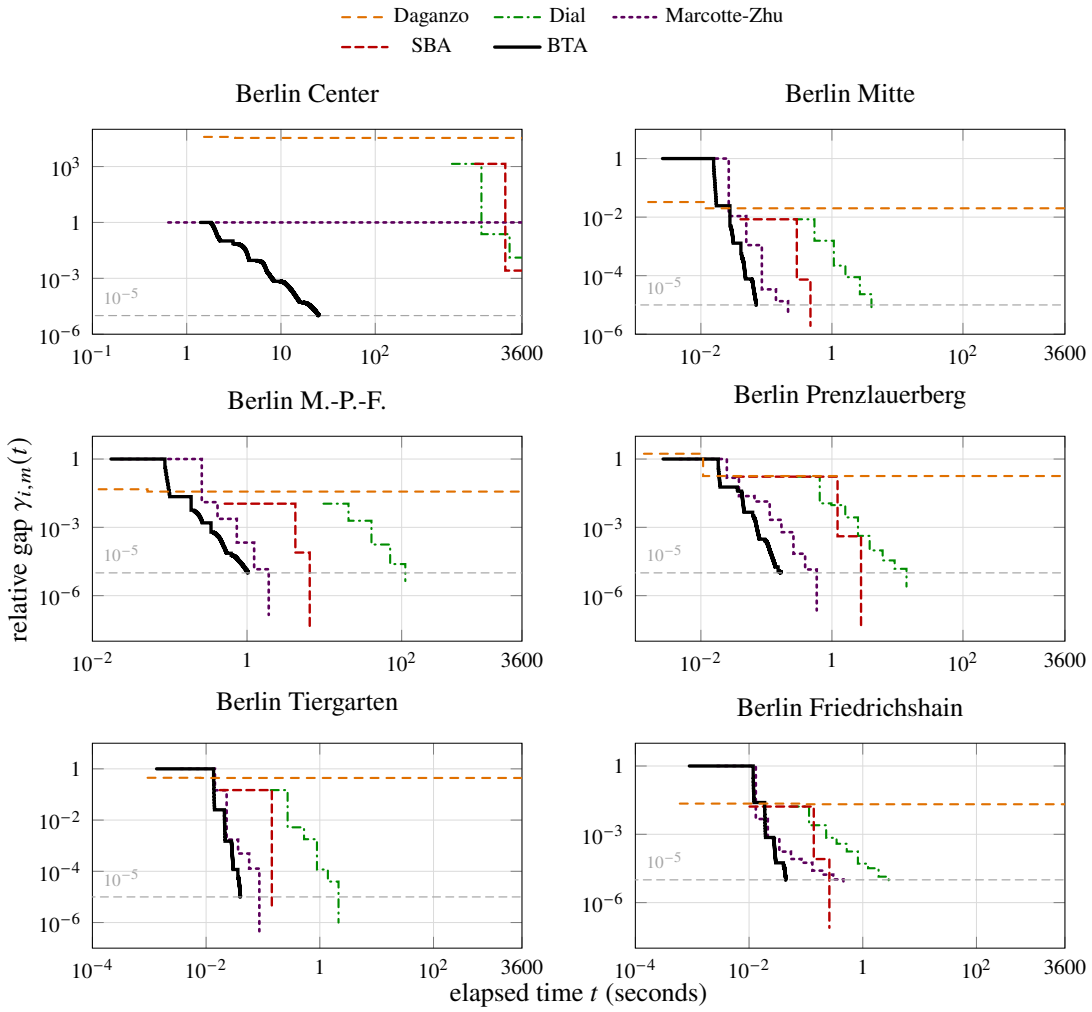


Figure EC.3 For each smooth instances i, the curves show the relative gaps $\gamma_{i,m}(t)$ of the five methods m as a function of computation time t. The dashed line shows the tolerance $\varepsilon_i = 10^{-5}$. Each curve stops at its first recorded point with $\gamma_{i,m}(t) \le \varepsilon_i$, or at T_i.

Table EC.4 compares the computational performance of Algorithm BTA and the method of Dafermos (1972) on nonsmooth instances i. If method m achieved relative gap $\gamma_{i,m}(t) \le \varepsilon_i = 10^{-3}$ for instance i at some time $t \le T_i$, where $T_i = 3600$ seconds for all instances except Goldcoast, Chicago Regional, and Philadelphia, which used $T_i = 86400$ seconds, then the table shows the solve time $t_{i,m}$ (in seconds), otherwise the table shows the final relative gap $\gamma_{i,m}(T_i)$ achieved by method m in parentheses. A summary of these results is given in Table 2.

Table EC.4 Computational performance on nonsmooth instances i. The table shows either the solve time $t_{i,m}$ (in seconds) for instance i if method m achieved relative gap $\gamma_{i,m}(t) \le \varepsilon_i = 10^{-3}$ at some time $t \le T_i$ (T_i = 24h for Goldcoast, Chicago Regional, and Philadelphia; T_i = 1h otherwise), or otherwise the final relative gap $\gamma_{i,m}(T_i)$ achieved by method m in parentheses.

Network	Dafermos	BTA
Braess	0.0545	0.016
Sioux Falls	0.031	0.191
Friedrichshain	0.060	15.671
Berlin Prenzlauerberg	0.315	39.797
Berlin Tiergarten	0.130	7.915
Berlin Mitte	0.220	33.597
Anaheim	0.670	140.964
Barcelona	26.931	2093.585
Winnipeg-Asym	26.739	42.728
Berlin M.-P.-F.	2.500	275.643
Winnipeg	21.447	84.343
Terrassa-Asym	25.665	280.418
Hessen-Asym	493.420	314.519
Goldcoast	3216.493	(5.32×10^{-3})
Chicago Regional	12385.983	20352.302
Berlin Center	420.221	2014.743
Philadelphia	36485.144	(1.65×10^{-2})

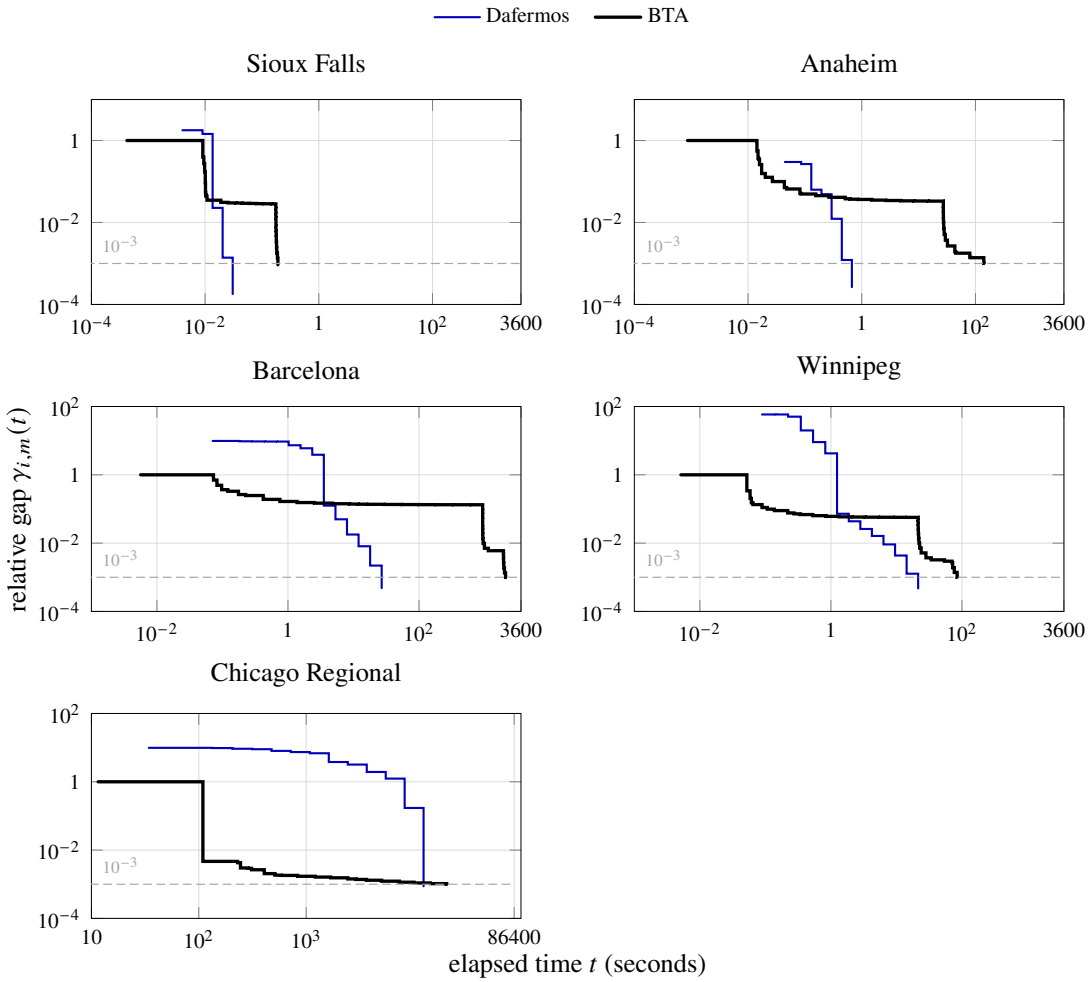


Figure EC.4 For nonsmooth instances i, the curves show the relative gaps $\gamma_{i,m}(t)$ of Dafermos and BTA as a function of computation time t. The dashed line shows the tolerance $\varepsilon_i = 10^{-3}$. Each curve stops at its first recorded point with $\gamma_{i,m}(t) \le \varepsilon_i$, or at T_i. For Chicago Regional, the BTA and Dafermos time limits are 12 and 24 hours, respectively.

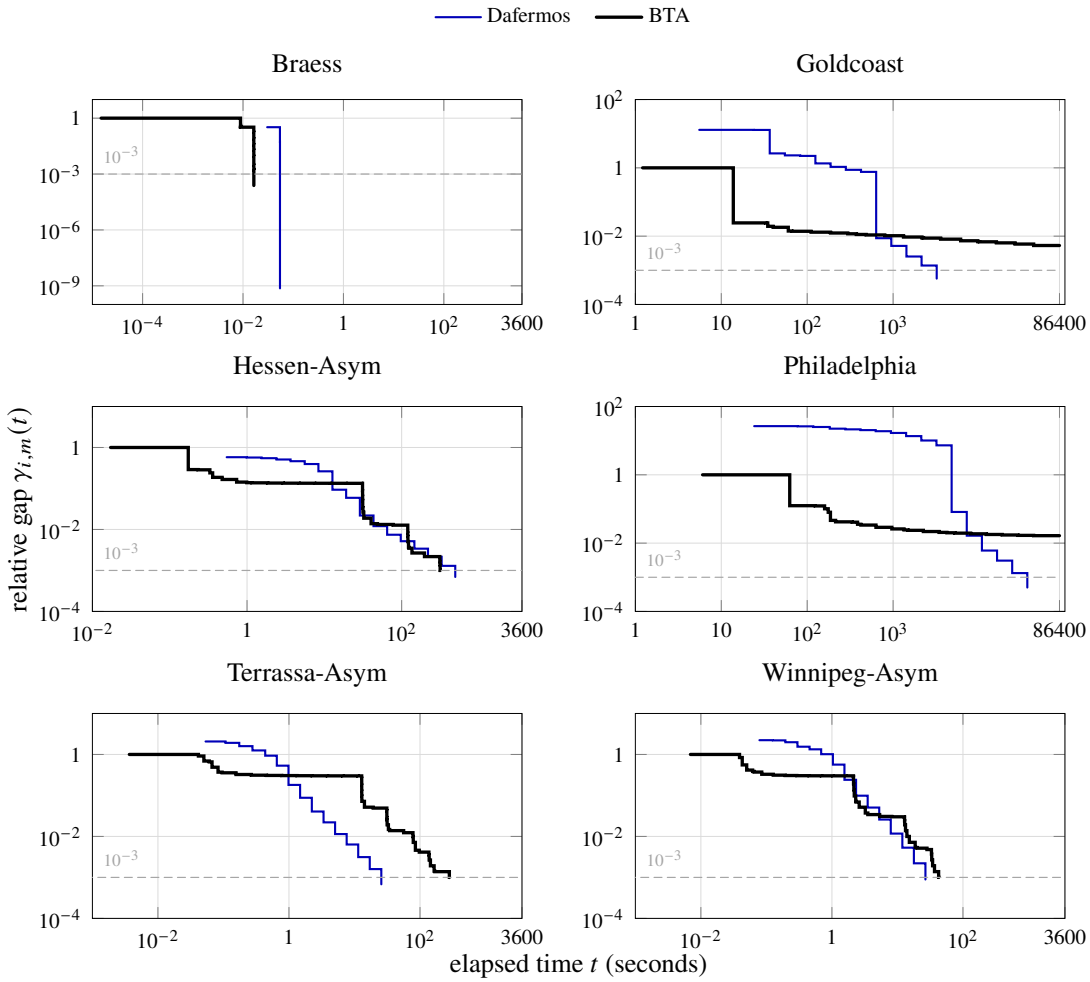


Figure EC.5 For nonsmooth instances i, the curves show the relative gaps $\gamma_{i,m}(t)$ of Dafermos and BTA as a function of computation time t. The dashed line shows the tolerance $\varepsilon_i = 10^{-3}$. Each curve stops at its first recorded point with $\gamma_{i,m}(t) \le \varepsilon_i$, or at T_i.

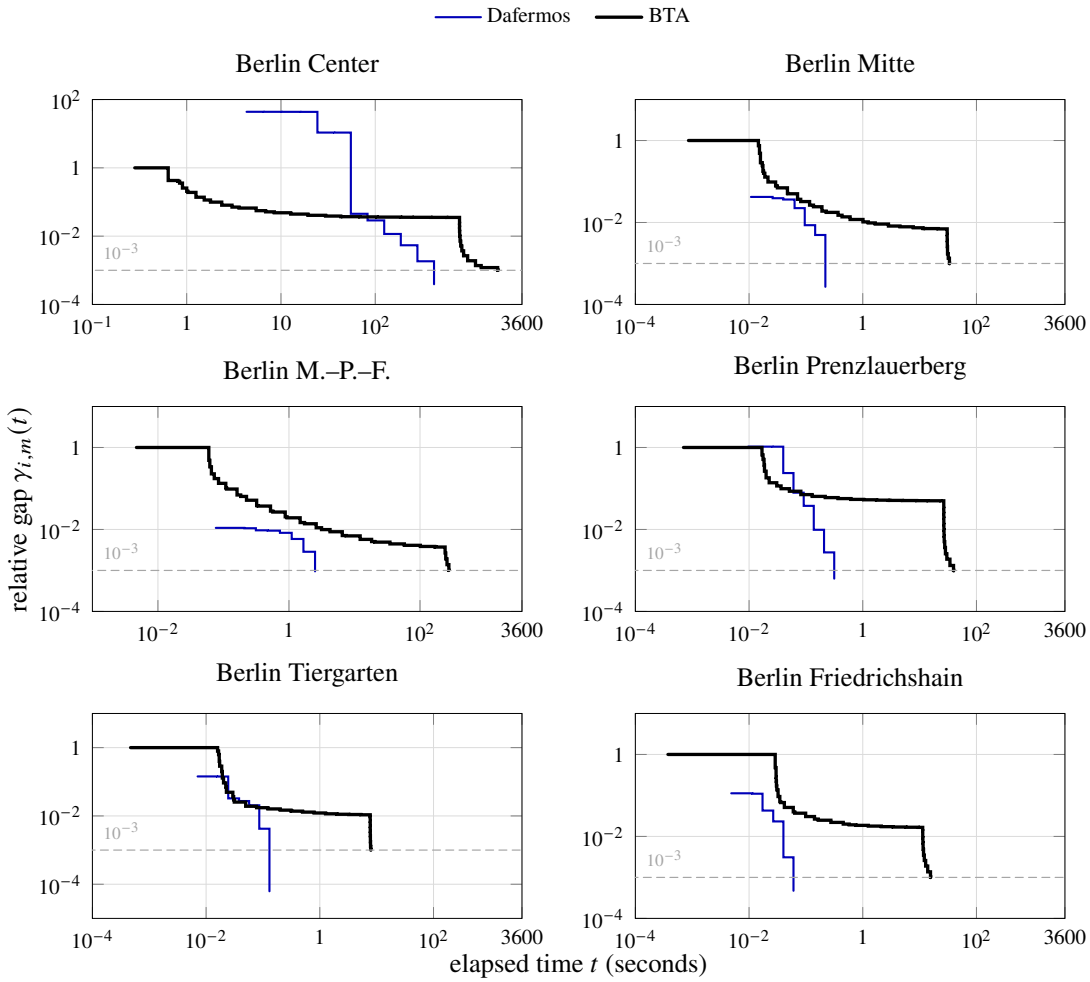


Figure EC.6 For nonsmooth instances i, the curves show the relative gaps $\gamma_{i,m}(t)$ of Dafermos and BTA as a function of computation time t. The dashed line shows the tolerance $\varepsilon_i = 10^{-3}$. Each curve stops at its first recorded point with $\gamma_{i,m}(t) \le \varepsilon_i$, or at T_i.

Path-support sparsity. As mentioned before, in typical networks only a small fraction of the paths from an origin to a destination carry positive flow. This observation motivated the design of an algorithm that optimizes the objective for (small) subsets $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$ of paths. A related statistic that is of interest is the number of paths generated by the algorithm ($\sum_{\pi\in\Pi} |\hat{\mathcal{R}}^\pi|$ when the algorithm stops) that carry positive flow in the final solution. We define a *used path* as a path r that has flow $q_r^{\pi*} > 10^{-6}$ in the final solution. Table EC.5 reports the number of generated paths, the maximum number of generated paths for any OD pair, the number of used paths, and the average number of used paths per OD pair, when the BTA algorithm stopped for Sioux Falls, Anaheim, Barcelona, and Winnipeg. For these instances, the fraction of paths generated by the algorithm that carry positive flow in the final solution is quite large, but the fraction tends to decrease as the network size increases.

Table EC.5 The number of paths generated by the algorithm ($\sum_{\pi\in\Pi} |\hat{\mathcal{R}}^\pi|$ when the algorithm stops), the maximum number of generated paths for an OD pair ($\max_{\pi\in\Pi} |\hat{\mathcal{R}}^\pi|$), the number of used paths, and the average number of used paths per OD pair, for BTA on four smooth instances.

Network	Generated paths	Maximum generated paths per OD pair	Used paths	Average used paths per OD pair
Sioux Falls	922	5	779	1.48
Anaheim	4513	22	3542	2.52
Barcelona	60435	59	31268	3.95
Winnipeg	57510	85	24642	5.67

Tolling schemes. For Capacity-UE, we computed a reference flow $\widehat{x}^{\mathrm{UE}}$ for the single criterion (time) equilibrium problem on each network with the same OD demands as the bicriteria instances. Recall that "arc capacities" c_a are parameters of the BPR travel-time functions. Then we assign monetary tolls

$$s_a = \begin{cases} \mathrm{round}_{0.01}\big(t_a(\widehat{x}_a^{\mathrm{UE}})\big) & \text{if } \widehat{x}_a^{\mathrm{UE}} > c_a \\ 0 & \text{if } \widehat{x}_a^{\mathrm{UE}} \le c_a \end{cases} \qquad a \in \mathcal{A},$$

where $\mathrm{round}_{0.01}$ denotes rounding to the nearest hundredth. Table EC.6 gives the results under the four tolling schemes of Section 5.3, including computation times, final gaps, used-path counts, and fraction of OD pairs with multiple paths with positive flow in equilibrium.

Table EC.6 Computation times, final gaps, used-path counts, and fraction of OD pairs with multiple used paths, for the four tolling schemes of Section 5.3. "Used" paths are paths with equilibrium flow above 10^{-6}.

	Smooth instances				Nonsmooth instances			
Toll scheme	Time	Gap	Used	Multi	Time	Gap	Used	Multi
Sioux Falls								
None	0.024	5.52×10^{-6}	734	0.265	0.475	7.11×10^{-4}	671	0.222
MC $\alpha_0=1$	0.021	9.13×10^{-6}	817	0.394	0.204	9.93×10^{-4}	975	0.570
MC $\alpha_0=18$	0.011	8.19×10^{-6}	692	0.286	0.286	9.62×10^{-4}	1176	0.697
Capacity-UE	0.013	9.55×10^{-6}	762	0.345	2.170	8.57×10^{-4}	904	0.540
Anaheim								
None	0.074	9.45×10^{-6}	1949	0.286	0.076	5.09×10^{-4}	2034	0.337
MC $\alpha_0=1$	0.270	9.86×10^{-6}	2757	0.583	0.532	9.73×10^{-4}	3631	0.764
MC $\alpha_0=18$	1.046	9.89×10^{-6}	3965	0.762	17.722	1.43×10^{-3}	7250	0.925
Capacity-UE	0.288	9.99×10^{-6}	2172	0.323	11.198	2.83×10^{-3}	3552	0.698
Berlin Tiergarten								
None	0.028	8.51×10^{-6}	863	0.222	0.180	6.49×10^{-4}	809	0.238
MC $\alpha_0=1$	0.044	9.45×10^{-6}	1144	0.484	0.259	9.93×10^{-4}	1565	0.618
MC $\alpha_0=18$	0.153	9.61×10^{-6}	1383	0.630	4.118	2.04×10^{-2}	2534	0.835
Capacity-UE	0.023	9.20×10^{-6}	810	0.247	1.740	9.40×10^{-4}	975	0.315
Winnipeg-Asym								
None	10.783	9.45×10^{-6}	19017	0.788	69.067	8.95×10^{-4}	14306	0.786
MC $\alpha_0=1$	5.919	9.84×10^{-6}	20061	0.827	16.850	9.72×10^{-4}	24380	0.882
MC $\alpha_0=18$	1.187	9.80×10^{-6}	14898	0.722	2.034	9.99×10^{-4}	19598	0.919
Capacity-UE	6.098	9.86×10^{-6}	19423	0.789	5.959	9.90×10^{-4}	14293	0.829