Bicriteria Traffic Assignment

Jacob M. Aguirre, Anton J. Kleywegt, and Renato D. C. Monteiro

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School of Industrial and Systems Engineering
Georgia Institute of Technology

Agenda

1. Bicriteria Traffic Assignment

2. Previous Formulations

3. A Finite-Dimensional Convex Formulation

4. From Optimality to Equilibrium

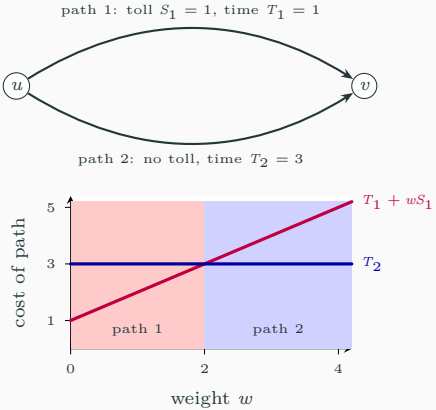
5. Algorithm BTA

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1.1 Traffic Assignment with Two Criteria

Observation:

- Many infinitesimal (nonatomic) users travel from their origins to their destinations in a congested network
- The travel time on each arc depends on the total flow on the arc
- In a Wardrop equilibrium, no user wants to change their chosen path [Wardrop, 1952]
- However, users care about cost (including tolls) as well as time, and **different users put different weights on these two criteria**
- A user with weight $w \geq 0$ on cost relative to time chooses a path that minimizes time $+\ w\times$ toll



1.2 Model

Setup:

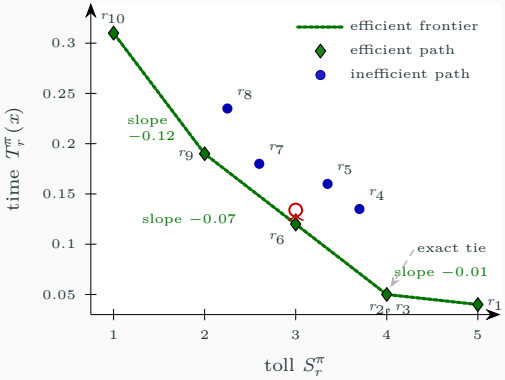
- Directed network $(\mathcal{N}, \mathcal{A})$; origin-destination (OD) pairs $\pi \in \Pi$ with demand rates $D^\pi > 0$
- Arc a: nondecreasing travel time $t_a(x_a)$ of the arc flow x_a, and flow-independent toll $s_a \geq 0$
- Path $r \in \mathcal{R}^\pi$: toll $S_r^\pi = \sum_{a \in \mathcal{A}_r^\pi} s_a$, travel time $T_r^\pi(x) = \sum_{a \in \mathcal{A}_r^\pi} t_a(x_a)$, and bicriteria cost

$$C_r^\pi(w, x) \quad = \quad T_r^\pi(x) + wS_r^\pi$$

- The weight w of the users of OD pair π has a cumulative distribution function G^π; $[\underline{w}^\pi, \overline{w}^\pi] \subset \mathbb{R}_+$ is the smallest interval that contains its support
- The toll does not depend on flow: this is necessary for the equilibrium conditions to coincide with the first-order conditions of an optimization problem [Marcotte et al., 1996], [Marcotte and Zhu, 1997]
- The paths of each OD pair are ordered so that $r \prec r'$ implies $S_r^\pi \geq S_{r'}^\pi$; unlike previous work, equal tolls $S_r^\pi = S_{r'}^\pi$ are allowed

1.3 Efficient Paths

- Given x, path r is **efficient** if it minimizes $C_r^\pi(w, x)$ for all w in an interval $[\underline{w}_r^\pi, \overline{w}_r^\pi]$
- Efficient paths $r^1, \ldots, r^k$ and thresholds $\underline{w}^\pi = w_0 \le \cdots \le w_k = \overline{w}^\pi$: path r^i is optimal on $[w_{i-1}, w_i]$
- The thresholds need not be unique: r_2 and r_3 are exactly tied
- r_6 is optimal only at $w = 0.07$; if G^π has an atom at 0.07, then r_6 can carry flow
- A small increase of $T^\pi_{r_6}(x)$ makes r_6 inefficient, and its flow can jump to zero



1.4 Objective

This work aims to compute equilibria of the bicriteria traffic assignment problem for arbitrary weight distributions G^π with bounded support, including mixtures of atoms, densities, and gaps in the support, by solving a finite-dimensional convex optimization problem.

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2.1 Finitely Many User Types

- [Dafermos, 1972]: finitely many user types $w = 1, \ldots, W$
  - symmetric Jacobians of the class costs yield an optimization formulation; for two criteria, $\partial c_a(w, x_a)/\partial x_a(w') = t'_a$ for all w, w', a symmetric positive semidefinite matrix
  - the algorithm moves flow from the most expensive used path of an OD pair to its shortest path
- [Dial, 1979]: flow-independent t_a and s_a; the algorithm computes the efficient paths and thresholds of each OD pair
- [Daganzo, 1983]: finitely many vehicle types and arc costs $t_a(x) + ws_a$ depending on all arc flows; existence by fixed point theorems, and a stochastic approximation algorithm
- These formulations require G^π to have **finite support**; every atom of G^π is a separate user class with its own path flows

2.2 Positive Density: Cumulative Path Flows

- [Leurent, 1993]: cumulative path flows $F^\pi_r = \sum_{r'=1}^{r} f^\pi_{r'}$, total flow F^π, and G^π with a positive density, so that $(G^\pi)^{-1}(q)$ is a singleton for every $q \in (0,1)$
- The equilibrium conditions are the first-order optimality conditions of

$$\min_{f \geq 0} \sum_{a \in \mathcal{A}} \int_0^{x_a} t_a(z)\,\mathrm{d}z + \sum_{\pi \in \Pi} F^\pi \sum_{r \in \mathcal{R}^\pi} S^\pi_r \int_{F^\pi_{r-1}/F^\pi}^{F^\pi_r/F^\pi} \left(G^\pi\right)^{-1}(q)\,\mathrm{d}q - \sum_{\pi \in \Pi} \int_0^{F^\pi} \left(D^\pi\right)^{-1}(q)\,\mathrm{d}q$$

  where D^π is a continuous decreasing demand function
- The objective is convex and the decision variables are finite-dimensional
- [Leurent, 1996]: comparison of three algorithms, and an augmented Lagrangian algorithm for arc capacity constraints $x_a \leq C_a$
- Limitation: a **positive density** is required

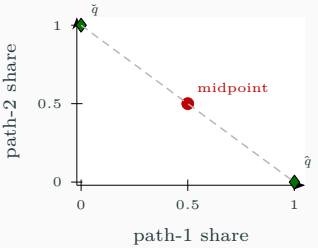
2.3 Function-Valued and Threshold Formulations

- [Marcotte and Zhu, 1996], [Marcotte et al., 1996]: weight-dependent arc flows $x_a^\pi(w)$ in $\mathcal{L}_2$
  - convex, but **infinite-dimensional**
  - the Frank–Wolfe subproblem is one parametric shortest path problem per OD pair
- [Dial, 1996], [Dial, 1997]: algorithms that alternate parametric shortest paths with a linear program over convex combinations of the generated flows
- [Marcotte, 1998]: thresholds w_r^π as decision variables
  - finite-dimensional, and the interval structure holds by construction
  - but the objective is **not convex** in w; a first-order point is nevertheless optimal
- [Xie et al., 2026]: the same threshold formulation; the Single-Boundary Adjustment (SBA) algorithm moves one threshold at a time with a line search

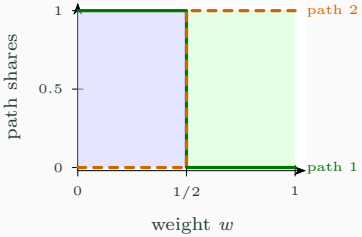
2.4 Convex Combinations and the Interval Structure

Example: one OD pair, two paths $1 \prec 2$, $D^\pi = 1$, and $w \sim \mathcal{U}[0,1]$

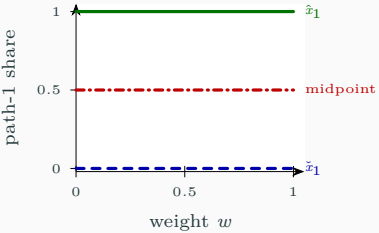
- Solution $\hat{}$ puts all flow on path 1, solution $\check{}$ puts all flow on path 2; take their midpoint



(a) aggregate formulations [Leurent, 1993], [Marcotte, 1998]



(b) weight-dependent flows implied by the midpoint in (a)



(c) function-valued formulation [Marcotte and Zhu, 1996]

The midpoint in (a) keeps the interval structure (users with $w < 1/2$ take path 1); the midpoint in (c) loses it, which is detrimental to computation

2.5 Summary of Formulations

Formulation	Decision variables	Convex	Distribution G^π
[Dafermos, 1972]	flows per user type	yes	finite support
[Leurent, 1993]	cumulative path flows	yes	positive density
[Marcotte and Zhu, 1996]	weight-dependent arc flows	yes	density
[Marcotte, 1998], [Xie et al., 2026]	thresholds w_r^π	no	positive density
This work	path fractions q_r^π	yes	any, bounded support

- Like [Leurent, 1993], the decision variables are finite-dimensional and the problem is convex
- Unlike all previous formulations, nothing is required of G^π besides bounded support

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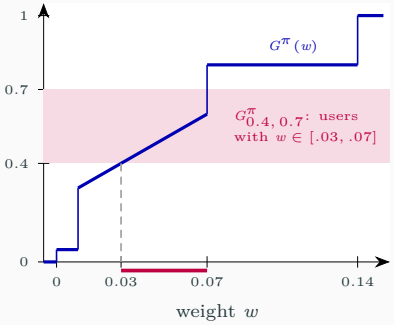
3.1 Decision Variables

Setup:

- q_r^π: fraction of the demand D^π that flows on path r, with q^π in the simplex $\Delta_{\mathcal{R}^\pi}$
- Cumulative fractions, linear in q^π: $p_r^\pi = L_r^\pi(q^\pi) = \sum_{r' \preceq r} q_{r'}^\pi$
- Arc flows $x_a(q) = \sum_{\pi \in \Pi} D^\pi \sum_{r \in \mathcal{R}_a^\pi} q_r^\pi$
- Truncated distribution, for $0 \le a \le b \le 1$:

$$G_{a,b}(y) \;=\; \max\{a, \min\{b, G(y)\}\}$$

a measure of mass $b - a$: the users between quantile levels a and b



3.2 Equilibrium Conditions

The pair (q, w) satisfies the **equilibrium conditions** if, for every $\pi \in \Pi$:

1. $q^\pi \in \Delta_{\mathcal{R}^\pi}$: each q^π is a probability distribution on the paths
2. $\underline{w}^\pi \le \cdots \le w^\pi_{r-1} \le w^\pi_r \le \cdots \le \overline{w}^\pi$: the thresholds are ordered
3. For each r with $q^\pi_r > 0$, each used path is user-optimal on its interval:

$$C^\pi_r(w, x(q)) \;\le\; C^\pi_{r'}(w, x(q)) \qquad \forall\, r' \in \mathcal{R}^\pi, \;\; \forall\, w \in [w^\pi_{r-1}, w^\pi_r]$$

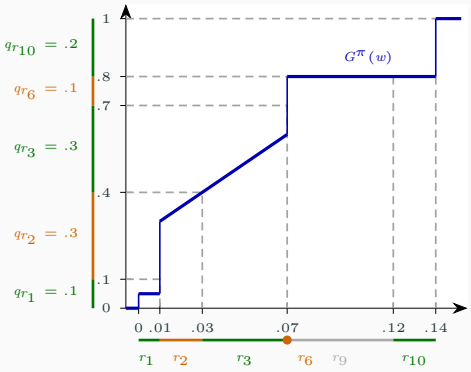
4. $w^\pi_r \in \big(G^\pi\big)^{-1}\big(L^\pi_r(q^\pi)\big)$: the thresholds and fractions are consistent, where the quantile map is

$$G^{-1}(p) \;=\; \{y \in \mathbb{R} \,:\, G(y') \le p \;\; \forall\, y' < y, \;\; G(y') \ge p \;\; \forall\, y' > y\}$$

The quantile set $G^{-1}(p)$ is a nonempty closed interval for every $p \in (0, 1)$; it has positive length when p is the level of a gap in the support

3.3 Example: Atoms, a Density, and a Gap

- G^π: atoms at 0, .01, .07, .14, a density on $[.01, .07)$, and a gap $(.07, .14)$
- $q^\pi = (.1, .3, .3, 0, 0, .1, 0, 0, 0, .2)$ and the thresholds shown form an equilibrium
- r_1, r_2 split the atom at .01; r_3, r_6 split the atom at .07
- r_6 carries flow on a singleton interval; r_9 carries none on $[.07, .12]$



3.4 Optimization Problem

For any path subsets $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$, with $p_r^\pi = L_r^\pi(q^\pi)$ regarded as linear functions of q:

- Travel-time term [Beckmann et al., 1956]: $\psi(q) \;=\; \sum_{a\in\mathcal{A}} \int_0^{x_a(q)} t_a(y)\,\mathrm{d}y$
- Toll term: the users between quantile levels p_{r-1}^π and p_r^π take path r and pay wS_r^π:

$$\phi(q) \;=\; \sum_{\pi\in\Pi} D^\pi \sum_{r\in\hat{\mathcal{R}}^\pi} S_r^\pi \int_{\mathbb{R}} y\,\mathrm{d}\,G^\pi_{p_{r-1}^\pi,\,p_r^\pi}(y)$$

- Optimization problem:

$$\min_q \left\{ \Upsilon(q) \;=\; \psi(q) + \phi(q) \;:\; q \in \textstyle\prod_{\pi\in\Pi} \Delta_{\hat{\mathcal{R}}^\pi} \right\}$$

- Finite-dimensional, and nothing is required of G^π besides bounded support
- A solution is q only: unlike previous formulations, its first-order conditions do not coincide with the equilibrium conditions; the thresholds w are recovered afterwards

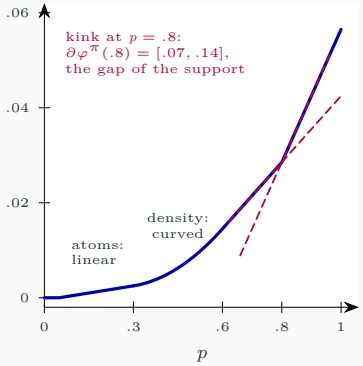
3.5 Convexity

- $\varphi^\pi(p) = \int_{\mathbb{R}} w \,\mathrm{d}\, G^\pi_{0,p}(w) = \int_0^p \left(G^\pi\right)^{-1}(u)\,\mathrm{d}u$ is convex and Lipschitz, with $\partial\varphi^\pi(p) = \left(G^\pi\right)^{-1}(p)$
- Atoms of G^π yield linear pieces of φ^π; gaps in the support yield kinks
- Summation by parts over the paths r with a successor $r+1$ in $\hat{\mathcal{R}}^\pi$:

$$\phi(q) = \sum_\pi D^\pi \sum_r \left(S^\pi_r - S^\pi_{r+1}\right) \varphi^\pi\left(L^\pi_r(q^\pi)\right) + \text{const}$$

  and $S^\pi_r \geq S^\pi_{r+1}$; hence ϕ is convex, and invariant to the order of equal-toll paths
- ψ is convex because each t_a is nondecreasing
- A gap in the support of G^π (for example, finitely many atoms) makes Υ **nonsmooth**

$\varphi^\pi(p)$ for the example G^π



3.6 Optimality Conditions

Lemma (each t_a continuous):

$q^* \in \prod_\pi \Delta_{\hat{\mathcal{R}}^\pi}$ is optimal if and only if there exist $w_r^{\pi *} \in \left(G^\pi\right)^{-1}\left(L_r^\pi(q^{\pi *})\right)$ and $\lambda^{\pi *}$ such that, for all $\pi \in \Pi$ and $r \in \hat{\mathcal{R}}^\pi$,

$$D^\pi \left[T_r^\pi(x(q^*)) + \sum_{r' \succeq r,\ r'+1 \in \hat{\mathcal{R}}^\pi} \left(S_{r'}^\pi - S_{r'+1}^\pi\right) w_{r'}^{\pi *} \right] + \lambda^{\pi *} \quad \geq \quad 0, \qquad \text{with equality if } q_r^{\pi *} > 0$$

- The bracket is a reduced path cost: travel time plus a telescoping toll term
- If all $w_{r'}^{\pi *}$ equal a common w, then the bracket equals $T_r^\pi(x(q^*)) + w(S_r^\pi - S_{\min}^\pi) = C_r^\pi(w, x(q^*)) - wS_{\min}^\pi$: used paths minimize the bicriteria cost
- The subgradients of ϕ are quantiles of G^π, that is, **candidate thresholds**
- Without continuity of t_a, the same holds with $t_a(x_a)$ replaced by some $\tau_a \in [t_a^-(x_a), t_a^+(x_a)]$

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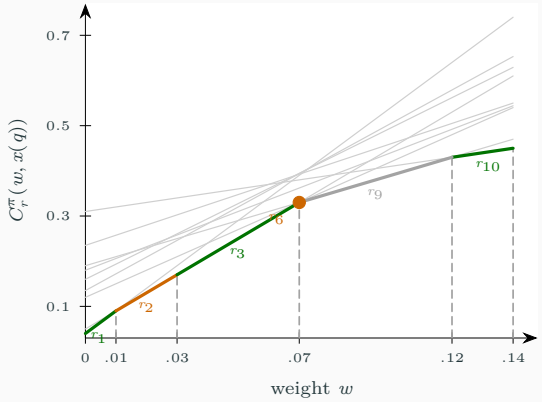
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4.1 Threshold Selection

Given an optimal q, for each OD pair π:

- Start at $\underline{w}^\pi$ with a path of least cost $C_r^\pi(\underline{w}^\pi, x(q))$
- Follow the lower envelope of the lines $w \mapsto T_r^\pi(x(q)) + wS_r^\pi$; kinks at .01, .07, .12
- Tied paths r_2, r_3 are separated by a quantile: $w_2^\pi = \max\{.01, \min(G^\pi)^{-1}(.4)\} = .03$
- Inefficient paths inherit the previous threshold
- r_6 gets the singleton $\{.07\}$; r_9 gets $[.07, .12]$ and no flow



4.2 Key Results

Lemma (each t_a continuous; q optimal for path sets $\hat{\mathcal{R}}^\pi$; w from threshold selection):

a) $\underline{w}^\pi = w_0^\pi \le w_1^\pi \le \cdots \le w^\pi_{\hat{R}^\pi} = \overline{w}^\pi$

b) every path with positive flow is efficient

c) $w_r^\pi \in \left(G^\pi\right)^{-1}\left(L_r^\pi(q^\pi)\right)$ for all r

d) every efficient path r_ℓ minimizes $T_r^\pi(x(q)) + wS_r^\pi$ over $r \in \hat{\mathcal{R}}^\pi$ for all $w \in [w^\pi_{r_{\ell-1}}, w^\pi_{r_\ell}]$

Theorem (optimality to equilibrium)

If each t_a is continuous and q is optimal with $\hat{\mathcal{R}}^\pi = \mathcal{R}^\pi$ for all π, then (q, w) satisfies the equilibrium conditions 1–4

- Continuity is needed: two parallel paths, $w = 0$, $t_1(x_1) = 1 + 2 \cdot \mathbb{1}\{x_1 \ge 1/2\}$, and $t_2 = 2$ admit no equilibrium
- Parametric shortest path thresholds ignore G^π: in the example they put the threshold between r_2 and r_3 at .07, not at the quantile .03

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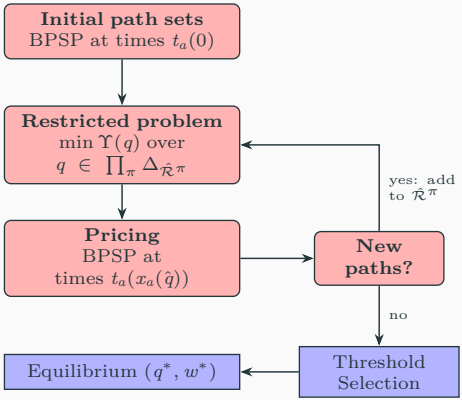
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5.1 Algorithm BTA

- In equilibrium only a small fraction of the paths of each OD pair carries flow
- BTA exploits convexity as well as this **sparsity**: it solves the convex problem over small path sets $\hat{\mathcal{R}}^\pi$
- Restricted problems: FISTA for smooth Υ [Sujanani and Monteiro, 2024]; two-cut adaptive proximal bundle method Ad-GPB for nonsmooth Υ [Monteiro and Zhang, 2024]
- New paths: one bicriteria parametric shortest path problem (BPSP) per origin
- Winnipeg: 57,510 generated paths for 4,345 OD pairs; 24,642 carry flow (5.67 per OD pair)



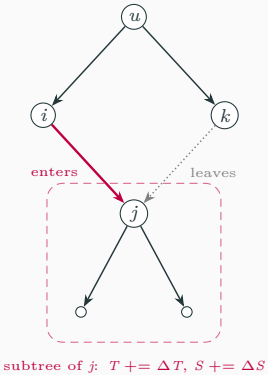
5.2 Bicriteria Parametric Shortest Paths

For each origin u [Karp and Orlin, 1981]:

- Compute a shortest path tree for arc costs $t_a + w s_a$ at $w = \underline{w}_u$, with time labels T_i and toll labels S_i
- Increase w to the next breakpoint

$$w' \;=\; \min_{(i,j)\notin \text{tree},\; S_i + s_{ij} < S_j} \frac{T_i + t_{ij} - T_j}{S_j - S_i - s_{ij}}$$

- Pivot: arc (i, j) enters the tree and replaces the tree arc into j; the labels of the subtree of j shift by ΔT and ΔS
- Each pivot adds an efficient path for every destination in the subtree of j
- Stop when $w' > \overline{w}_u$, or when no arc reduces a toll label



5.3 Convergence of BTA

Lemma (optimality certificate):
If every parametric shortest path at $x(\hat{q})$ is already in $\hat{\mathcal{R}}^\pi$ for all π, then $\hat{q}$ extended by zeros, together with w^* from threshold selection and an explicit multiplier λ^*, satisfies the optimality conditions for the complete path sets

Theorem

BTA returns $\hat{q}$ in $O\big(\sum_{\pi\in\Pi}|\hat{\mathcal{R}}^\pi|\big)$ steps; $\hat{q}$ extended by zeros is optimal for the complete path sets, and with the thresholds w^* it is an equilibrium of the bicriteria traffic assignment problem

Example (a ten-path network with the data of the efficient-path example):

- First path set $\{r_1, r_3, r_6, r_9, r_{10}\}$; restricted solution $\hat{q} = \left(\frac{13}{90}, \frac{13}{30}, \frac{1}{10}, \frac{11}{90}, \frac{1}{5}\right)$
- At $x(\hat{q})$: $T^\pi_{r_2} = \frac{1}{20} < \frac{49}{900} = T^\pi_{r_3}$, and pricing adds r_2
- Second restricted solution $q^* = (.1, .3, .3, 0, 0, .1, 0, 0, 0, .2)$; pricing adds no path, and BTA stops

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6.1 Experimental Setup

- 17 benchmark networks, from Braess (4 nodes) to Chicago Regional (12,979 nodes, 39,018 links, 3,136,441 OD records)
- BPR travel times $t_a(x_a) = t_a^0\big(1 + \alpha_a(x_a/c_a)^{\beta_a}\big)$; marginal-cost tolls $s_a = \alpha_0\, x_a^{\mathrm{SO}}\, t_a'(x_a^{\mathrm{SO}})$ rounded to 0.01, with $\alpha_0 = 1$
- **Smooth** instances: $w \sim \mathcal{U}[6, 30]$; **nonsmooth** instances: equal atoms at $w = 0, 2.5, 5, 7.5, 10$
- Methods compared:
  - smooth: stochastic approximation [Daganzo, 1983], T2-ETA [Dial, 1996], parametric shortest path descent with exact line search [Marcotte et al., 1996], SBA [Xie et al., 2026]
  - nonsmooth: path correction for finitely many classes [Dafermos, 1972]
- All methods in C11 with POSIX threads, except SBA (the authors' C++14 code)

6.2 Benchmarking Metrics

- Best objective value of method m on instance i by time t:
  $V_{i,m}(t) = \min_{t' \le t} \Upsilon_i(q_{i,m}(t'))$
- Convexity of Υ_i implies, for every feasible q and $g \in \partial\Upsilon_i(q)$, the lower bound

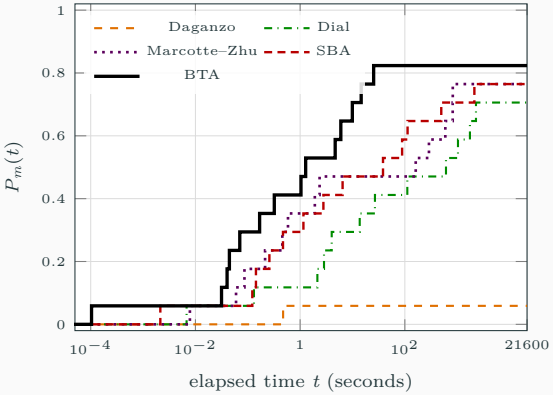
$$\Upsilon_i^* \quad \ge \quad \Upsilon_i(q) + \min\left\{ \langle g, y - q \rangle : y \in \prod_\pi \Delta_{\mathcal{R}^\pi} \right\}$$

  L_i: the largest such bound over the iterates of all methods
- Relative gap $\gamma_{i,m}(t) = \dfrac{V_{i,m}(t) - L_i}{\Upsilon_i(q_i^0) - L_i}$, where q_i^0 is the initial point of BTA
- Tolerance $\varepsilon_i = 10^{-5}$ (smooth) and 10^{-3} (nonsmooth); time limit $T_i = 1$h, except for Goldcoast, Chicago Regional, and Philadelphia (6h smooth, 12h nonsmooth)
- $P_m(t)$: fraction of instances solved by time t; g_m: shifted geometric mean of the capped solve times $t_{i,m}$; $r_m = g_m / \min_{m'} g_{m'}$; h_m: fraction of instances on which m is fastest

6.3 Smooth Instances

Method	g_m	r_m	h_m	solved
Daganzo	3120.898	250.622	0.000	1/17
Dial	196.485	15.779	0.000	12/17
Marcotte–Zhu	68.737	5.520	0.000	13/17
SBA	59.983	4.817	0.000	13/17
BTA	**12.453**	**1.000**	**0.824**	**14/17**

- BTA is the fastest on all 14 instances solved by some method
- No method reaches 10^{-5} on Goldcoast, Chicago Regional, or Philadelphia within 6h
- BTA has the smallest shifted geometric mean, 4.8× smaller than SBA



6.4 Smooth Instances: Solve Times

Network	Daganzo	Dial	Marcotte–Zhu	SBA	BTA
Braess	(2.38×10^{-1})	0.00678	0.00784	0.00214	$< \mathbf{0.001}$
Sioux Falls	0.473	0.131	0.0597	0.122	**0.032**
Friedrichshain	(2.09×10^{-2})	2.86	0.457	0.259	**0.045**
Berlin Prenzlauerberg	(1.79×10^{-1})	13.7	0.586	2.78	**0.168**
Berlin Tiergarten	(4.41×10^{-1})	2.13	0.0864	0.143	**0.040**
Berlin Mitte	(2.00×10^{-2})	4.01	0.214	0.470	**0.070**
Anaheim	(2.61×10^{-1})	26.9	2.35	1.16	**0.321**
Barcelona	(24.1)	1750	289	38.3	**5.987**
Winnipeg-Asym	(6.22×10^{-2})	611	161	113	**1.275**
Berlin M.-P.-F.	(3.72×10^{-2})	112	1.91	6.47	**1.038**
Winnipeg	(33.5)	2280	823	88.7	**4.673**
Terrassa-Asym	(3.59×10^{-1})	1030	593	496	**9.935**
Hessen-Asym	(3.59×10^{-2})	(1.94×10^{-5})	820	2130	**14.657**
Goldcoast	(112)	(4.71)	(1.37×10^{-2})	$(\mathbf{1.01 \times 10^{-2}})$	(1.09×10^{-2})
Chicago Regional	(3.28×10^{3})	—	(8.67×10^{-1})	(1.85×10^{4})	$(\mathbf{2.03 \times 10^{-1}})$
Berlin Center	(3.44×10^{4})	(1.29×10^{-2})	(1.00)	(2.60×10^{-3})	**25.352**
Philadelphia	(2.21)	—	(1.00)	(24.7)	$(\mathbf{5.06 \times 10^{-4}})$

Solve time $t_{i,m}$ in seconds to reach $\gamma_{i,m}(t) \leq 10^{-5}$; otherwise the final gap $\gamma_{i,m}(T_i)$ in parentheses; "—": no feasible solution saved within T_i. Bold: fastest time, or smallest final gap.

6.5 Nonsmooth Instances

Method	g_m	r_m	h_m	solved
Dafermos	25.641	1.000	0.882	17/17
BTA	204.737	7.985	0.118	15/17

- [Dafermos, 1972] is designed for finitely many classes, that is, for this setting (one class per atom)
- Dafermos is faster on 15 of the 17 instances; BTA is faster on Braess and Hessen-Asym
- BTA reaches 10^{-3} on all instances except Goldcoast and Philadelphia
- BTA solves the restricted problems to the final accuracy, instead of stopping earlier to generate new paths

Network	Dafermos	BTA
Braess	0.0545	**0.016**
Sioux Falls	**0.031**	0.191
Friedrichshain	**0.060**	15.671
Berlin Prenzlauerberg	**0.315**	39.797
Berlin Tiergarten	**0.130**	7.915
Berlin Mitte	**0.220**	33.597
Anaheim	**0.670**	140.964
Barcelona	**26.931**	2093.585
Winnipeg-Asym	**26.739**	42.728
Berlin M.-P.-F.	**2.500**	275.643
Winnipeg	**21.447**	84.343
Terrassa-Asym	**25.665**	280.418
Hessen-Asym	493.420	**314.519**
Goldcoast	**3216.493**	(5.32×10^{-3})
Chicago Regional	**12385.983**	20352.302
Berlin Center	**420.221**	2014.743
Philadelphia	**36485.144**	(1.65×10^{-2})

Solve time (s) to $\gamma \leq 10^{-3}$, or final gap in parentheses

6.6 Tolls and Path Diversity

Fraction of OD pairs with positive equilibrium flow on more than one path:

Network	Instance	No toll	MC $\alpha_0 = 1$	MC $\alpha_0 = 18$	Capacity-UE
Sioux Falls	smooth	0.265	0.394	0.286	0.345
	nonsmooth	0.222	0.570	0.697	0.540
Anaheim	smooth	0.286	0.583	0.762	0.323
	nonsmooth	0.337	0.764	0.925	0.698
Berlin Tiergarten	smooth	0.222	0.484	0.630	0.247
	nonsmooth	0.238	0.618	0.835	0.315
Winnipeg-Asym	smooth	0.788	0.827	0.722	0.789
	nonsmooth	0.786	0.882	0.919	0.829

- No toll is the single-criterion baseline; Capacity-UE tolls the arcs whose single-criterion equilibrium flow exceeds capacity
- With a second criterion, more OD pairs split their flow over several paths, and in most (but not all) cases the fraction increases with the tolls

Concluding Remarks

- A finite-dimensional convex formulation in the path fractions q for weight distributions with bounded support, including atoms, densities, and gaps
- Threshold selection converts any optimal q into an equilibrium (q, w), also when tolls or times are tied
- BTA alternates restricted convex problems over small path sets with bicriteria parametric shortest paths; the zero extension certifies optimality for the complete path sets
- Smooth instances: BTA is the fastest method on all 14 instances solved by some method
- Finite support: the method of [Dafermos, 1972] is faster; stopping the restricted problems early to generate paths sooner requires a different algorithm, and is an interesting topic for future research

Some Related Works

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