Bicriteria Traffic Assignment

Jacob M. Aguirre, Anton J. Kleywegt, and Renato D. C. Monteiro

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School of Industrial and Systems Engineering
Georgia Institute of Technology

Agenda

1. Bicriteria Traffic Assignment
2. Previous Formulations
3. A Finite-Dimensional Convex Formulation
4. From Optimality to Equilibrium
5. Algorithm BTA
6. Numerical Results

1.1 Traffic Assignment with Two Criteria

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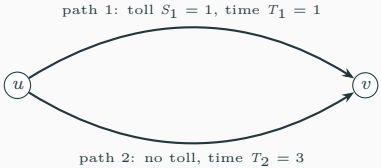

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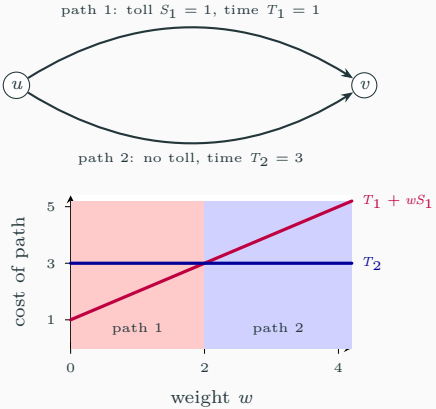
- However, users care about cost (including tolls) as well as time, and **different users put different weights on these two criteria**
- A user with weight $w \geq 0$ on cost relative to time chooses a path that minimizes time $+ w\times$ toll



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- The paths of each OD pair are ordered so that $r \prec r'$ implies $S_r^\pi \geq S_{r'}^\pi$; unlike previous work, equal tolls $S_r^\pi = S_{r'}^\pi$ are allowed

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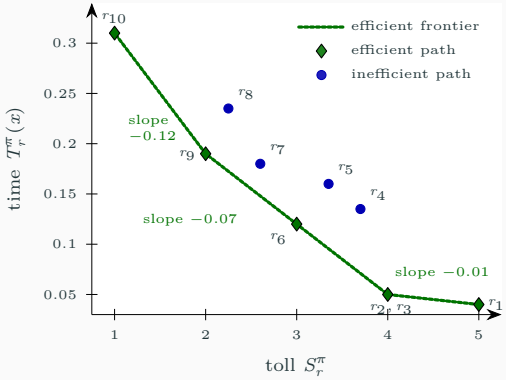

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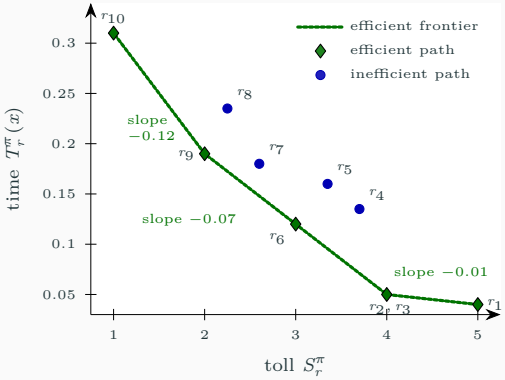

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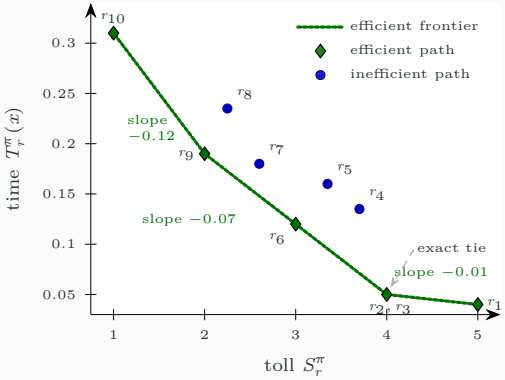

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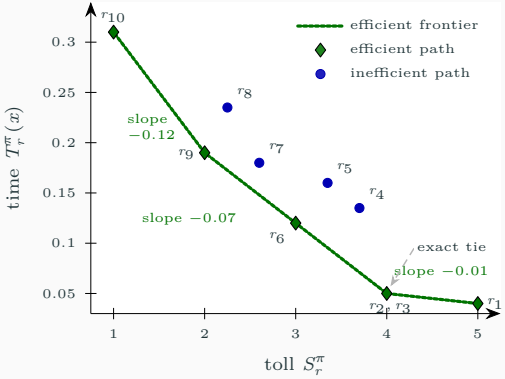

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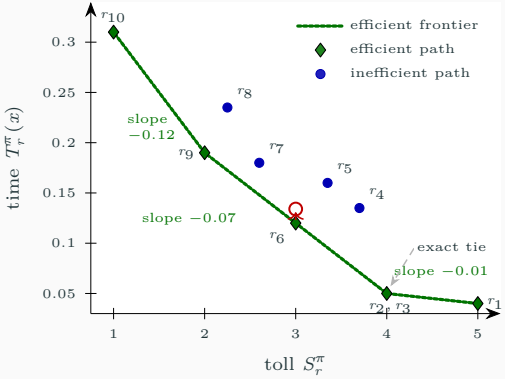
- A small increase of $T_{r_6}^\pi(x)$ makes r_6 inefficient, and its flow can jump to zero



1.4 Objective

This work aims to compute equilibria of the bicriteria traffic assignment problem for arbitrary weight distributions G^π with bounded support, including mixtures of atoms, densities, and gaps in the support, by solving a finite-dimensional convex optimization problem.

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- These formulations require G^π to have **finite support**; every atom of G^π is a separate user class with its own path flows

2.2 Positive Density: Cumulative Path Flows

- [Leurent, 1993]: cumulative path flows $F^{\pi}_{r} = \sum_{r'=1}^{r} f^{\pi}_{r'}$, total flow F^{π}, and G^{π} with a positive density, so that $(G^{\pi})^{-1}(q)$ is a singleton for every $q \in (0, 1)$

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- Limitation: a **positive density** is required

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- [Xie et al., 2026]: the same threshold formulation; the Single-Boundary Adjustment (SBA) algorithm moves one threshold at a time with a line search

2.4 Convex Combinations and the Interval Structure

Example: one OD pair, two paths $1 \prec 2$, $D^\pi = 1$, and $w \sim \mathcal{U}[0,1]$

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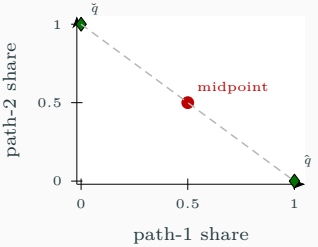

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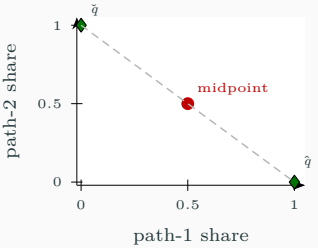

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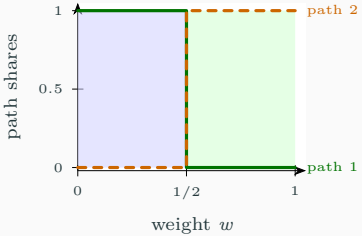

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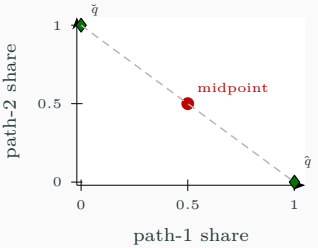

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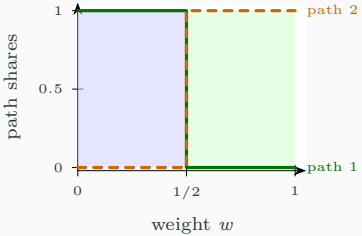

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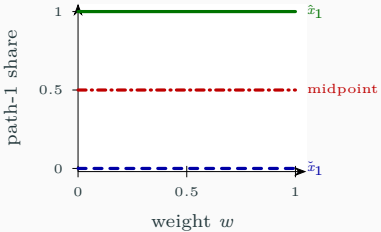

(c) function-valued formulation [Marcotte and Zhu, 1996]

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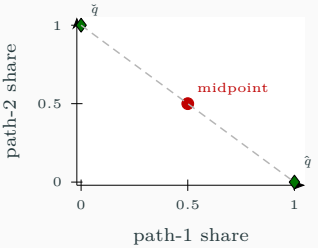
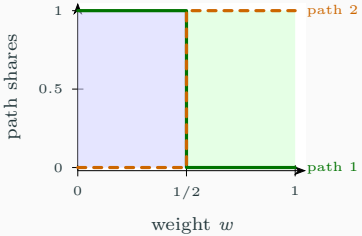
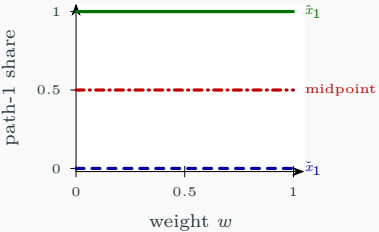


(a) aggregate formulations [Leurent, 1993], [Marcotte, 1998]



(b) weight-dependent flows implied by the midpoint in (a)



(c) function-valued formulation [Marcotte and Zhu, 1996]

The midpoint in (a) keeps the interval structure (users with $w < 1/2$ take path 1); the midpoint in (c) loses it, which is detrimental to computation

2.5 Summary of Formulations

Formulation	Decision variables	Convex	Distribution G^π
[Dafermos, 1972]	flows per user type	yes	finite support
[Leurent, 1993]	cumulative path flows	yes	positive density
[Marcotte and Zhu, 1996]	weight-dependent arc flows	yes	density
[Marcotte, 1998], [Xie et al., 2026]	thresholds w_r^π	no	positive density

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[Marcotte, 1998], [Xie et al., 2026]	thresholds w_r^π	no	positive density
This work	path fractions q_r^π	yes	any, bounded support

- Like [Leurent, 1993], the decision variables are finite-dimensional and the problem is convex
- Unlike all previous formulations, nothing is required of G^π besides bounded support

Agenda

1. Bicriteria Traffic Assignment

2. Previous Formulations

3. A Finite-Dimensional Convex Formulation

4. From Optimality to Equilibrium

5. Algorithm BTA

6. Numerical Results

3.1 Decision Variables

Setup:

- q_r^π: fraction of the demand D^π that flows on path r, with q^π in the simplex $\Delta_{\mathcal{R}^\pi}$

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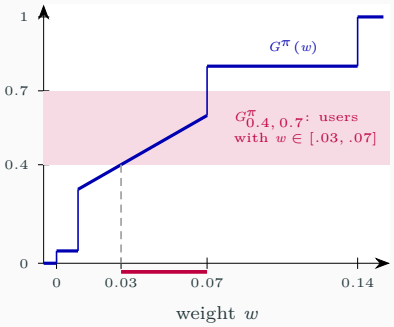
- Truncated distribution, for $0 \le a \le b \le 1$:

$$G_{a,b}(y) \;=\; \max\{a, \min\{b, G(y)\}\}$$

a measure of mass $b - a$: the users between quantile levels a and b



3.2 Equilibrium Conditions

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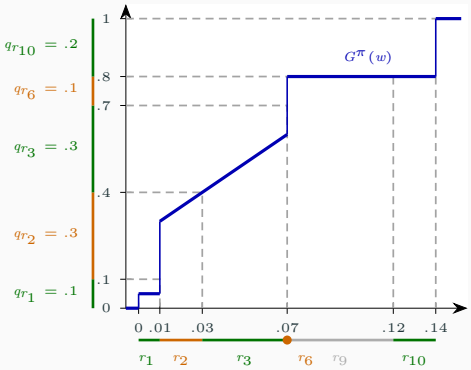
4. $w^\pi_r \in \big(G^\pi\big)^{-1}\big(L^\pi_r(q^\pi)\big)$: the thresholds and fractions are consistent, where the quantile map is

$$G^{-1}(p) \;=\; \{y \in \mathbb{R} \,:\, G(y') \le p \;\; \forall\, y' < y, \;\; G(y') \ge p \;\; \forall\, y' > y\}$$

The quantile set $G^{-1}(p)$ is a nonempty closed interval for every $p \in (0, 1)$; it has positive length when p is the level of a gap in the support

3.3 Example: Atoms, a Density, and a Gap

- G^π: atoms at 0, .01, .07, .14, a density on $[.01, .07)$, and a gap $(.07, .14)$



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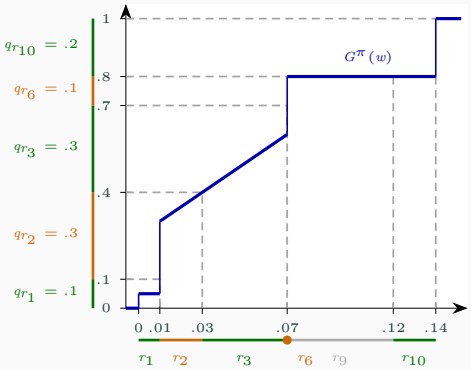

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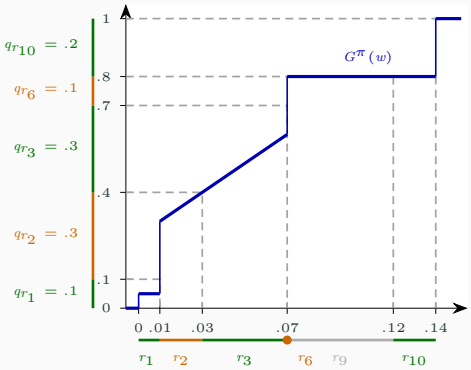

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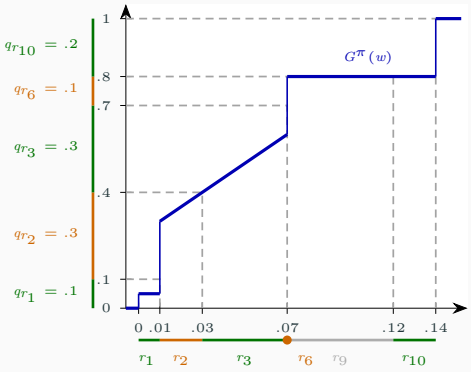
- r_6 carries flow on a singleton interval; r_9 carries none on $[.07, .12]$



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For any path subsets $\hat{\mathcal{R}}^\pi \subset \mathcal{R}^\pi$, with $p_r^\pi = L_r^\pi(q^\pi)$ regarded as linear functions of q:

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- A solution is q only: unlike previous formulations, its first-order conditions do not coincide with the equilibrium conditions; the thresholds w are recovered afterwards

3.5 Convexity

- $\varphi^\pi(p) = \int_{\mathbb{R}} w \, \mathrm{d} G^\pi_{0,p}(w) = \int_0^p \left(G^\pi\right)^{-1}(u) \, \mathrm{d}u$ is convex and Lipschitz, with $\partial\varphi^\pi(p) = \left(G^\pi\right)^{-1}(p)$

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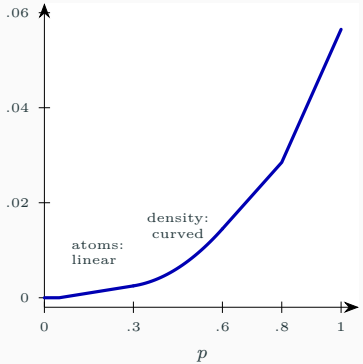

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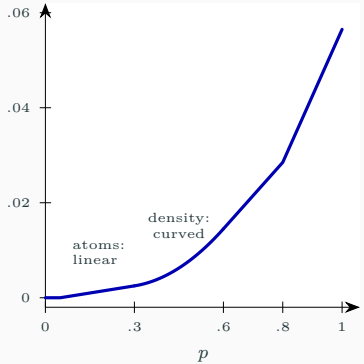

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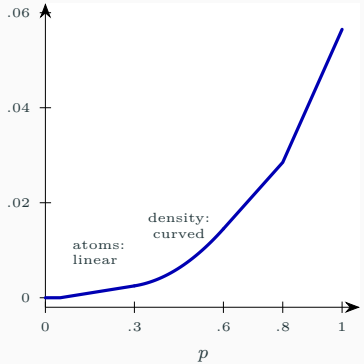

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- A gap in the support of G^π (for example, finitely many atoms) makes Υ **nonsmooth**

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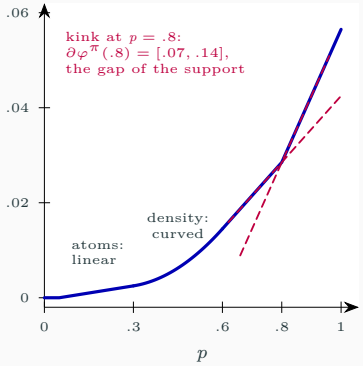

3.6 Optimality Conditions

Lemma (each t_a continuous):

$q^* \in \prod_\pi \Delta_{\hat{\mathcal{R}}^\pi}$ is optimal if and only if there exist $w_r^{\pi *} \in \left(G^\pi\right)^{-1}\left(L_r^\pi(q^{\pi *})\right)$ and $\lambda^{\pi *}$ such that, for all $\pi \in \Pi$ and $r \in \hat{\mathcal{R}}^\pi$,

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- Without continuity of t_a, the same holds with $t_a(x_a)$ replaced by some $\tau_a \in [t_a^-(x_a), t_a^+(x_a)]$

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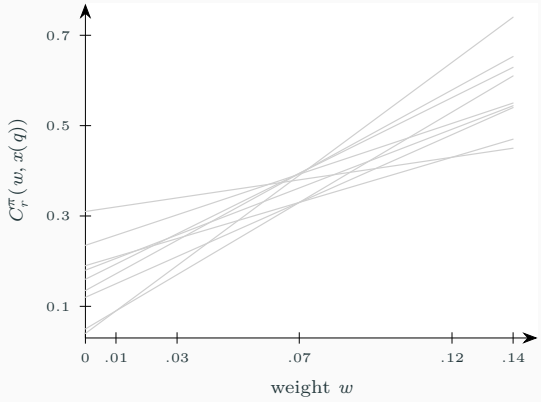

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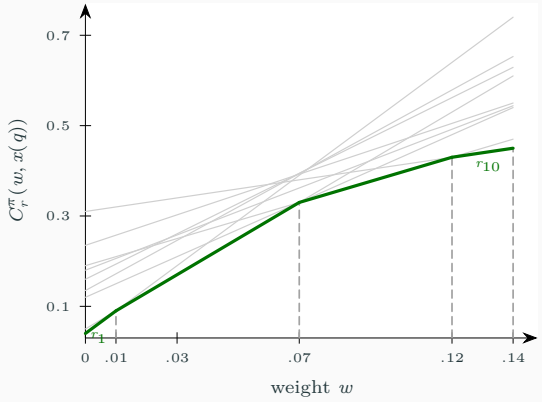

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- Start at $\underline{w}^\pi$ with a path of least cost $C_r^\pi(\underline{w}^\pi, x(q))$
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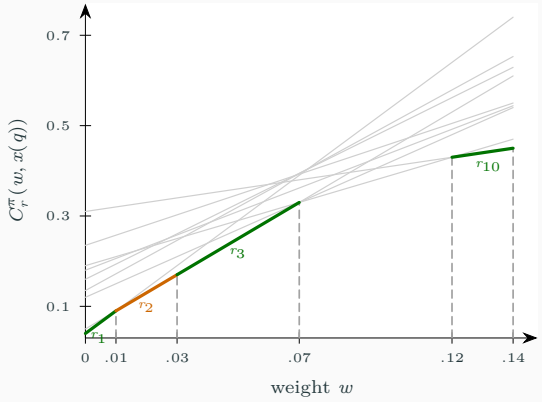
- Tied paths r_2, r_3 are separated by a quantile: $w_2^\pi = \max\{.01, \min(G^\pi)^{-1}(.4)\} = .03$



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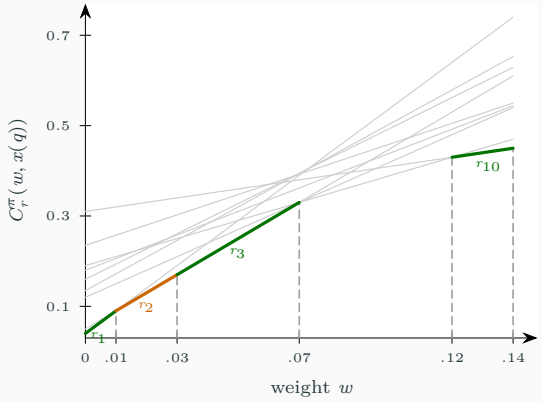

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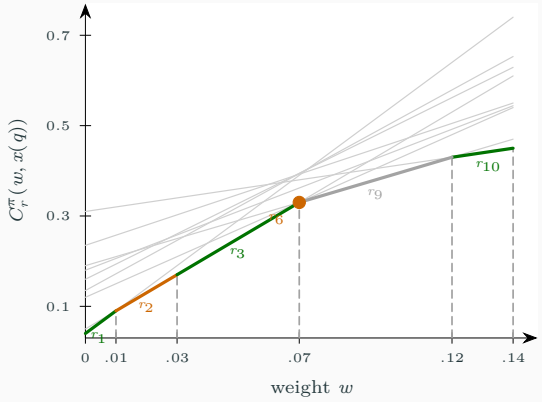

4.2 Key Results

Lemma (each t_a continuous; q optimal for path sets $\hat{\mathcal{R}}^\pi$; w from threshold selection):

a) $\underline{w}^\pi = w_0^\pi \le w_1^\pi \le \cdots \le w_{\hat{R}^\pi}^\pi = \overline{w}^\pi$

b) every path with positive flow is efficient

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- Parametric shortest path thresholds ignore G^π: in the example they put the threshold between r_2 and r_3 at .07, not at the quantile .03

Agenda

1. Bicriteria Traffic Assignment

2. Previous Formulations

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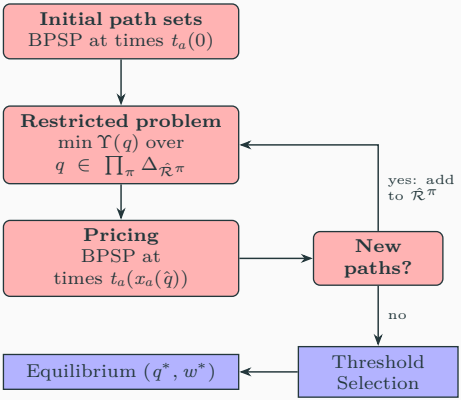
6. Numerical Results

5.1 Algorithm BTA

- In equilibrium only a small fraction of the paths of each OD pair carries flow



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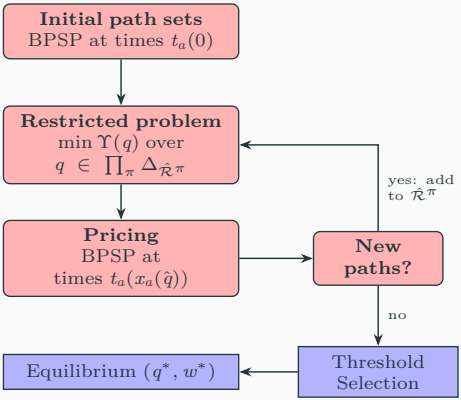

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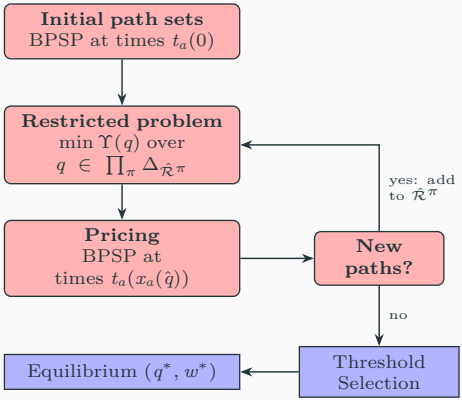

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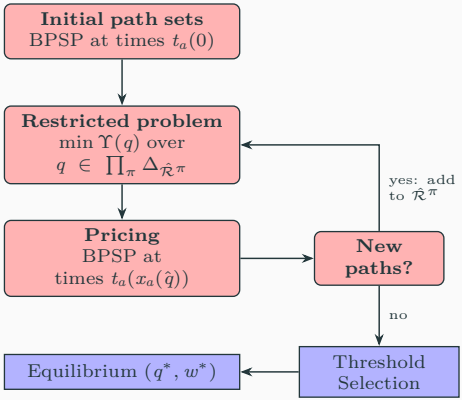

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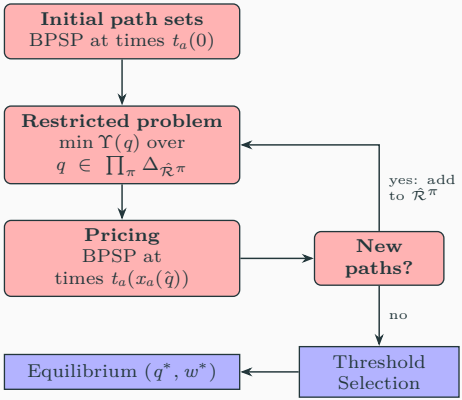

5.2 Bicriteria Parametric Shortest Paths

For each origin u [Karp and Orlin, 1981]:

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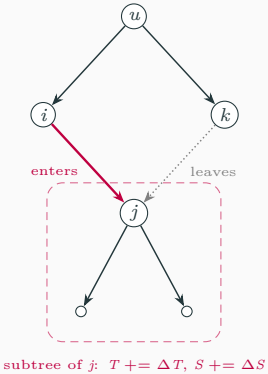

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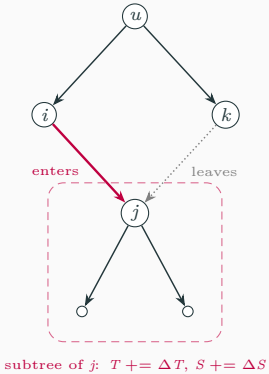

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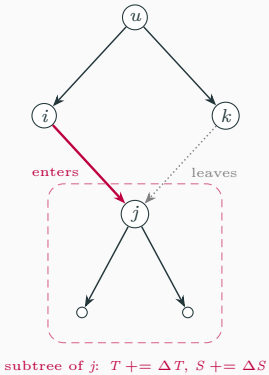

5.3 Convergence of BTA

Lemma (optimality certificate):
If every parametric shortest path at $x(\hat{q})$ is already in $\hat{\mathcal{R}}^{\pi}$ for all π, then $\hat{q}$ extended by zeros, together with w^* from threshold selection and an explicit multiplier λ^*, satisfies the optimality conditions for the complete path sets

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Example (a ten-path network with the data of the efficient-path example):

- First path set $\{r_1, r_3, r_6, r_9, r_{10}\}$; restricted solution $\hat{q} = \left(\frac{13}{90}, \frac{13}{30}, \frac{1}{10}, \frac{11}{90}, \frac{1}{5}\right)$
- At $x(\hat{q})$: $T^\pi_{r_2} = \frac{1}{20} < \frac{49}{900} = T^\pi_{r_3}$, and pricing adds r_2
- Second restricted solution $q^* = (.1, .3, .3, 0, 0, .1, 0, 0, 0, .2)$; pricing adds no path, and BTA stops

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6.2 Benchmarking Metrics

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- $P_m(t)$: fraction of instances solved by time t; g_m: shifted geometric mean of the capped solve times $t_{i,m}$; $r_m = g_m / \min_{m'} g_{m'}$; h_m: fraction of instances on which m is fastest

6.3 Smooth Instances

Method	g_m	r_m	h_m	solved
Daganzo	3120.898	250.622	0.000	1/17
Dial	196.485	15.779	0.000	12/17
Marcotte–Zhu	68.737	5.520	0.000	13/17
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BTA	**12.453**	**1.000**	**0.824**	**14/17**

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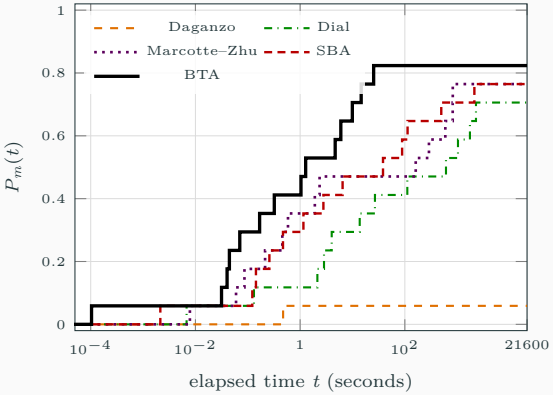

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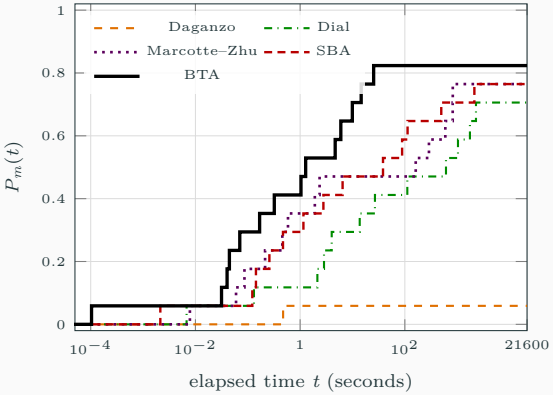
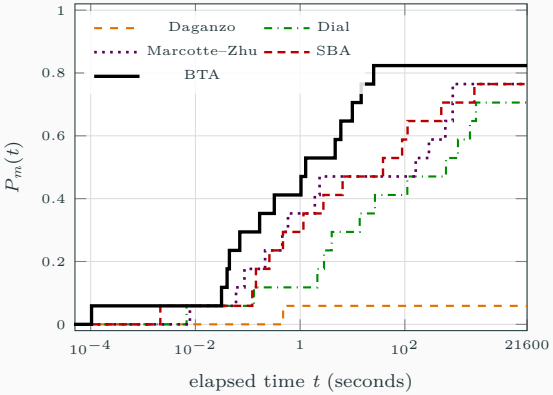
- No method reaches 10^{-5} on Goldcoast, Chicago Regional, or Philadelphia within 6h



6.3 Smooth Instances

Method	g_m	r_m	h_m	solved
Daganzo	3120.898	250.622	0.000	1/17
Dial	196.485	15.779	0.000	12/17
Marcotte–Zhu	68.737	5.520	0.000	13/17
SBA	59.983	4.817	0.000	13/17
BTA	**12.453**	**1.000**	**0.824**	**14/17**

- BTA is the fastest on all 14 instances solved by some method
- No method reaches 10^{-5} on Goldcoast, Chicago Regional, or Philadelphia within 6h
- BTA has the smallest shifted geometric mean, 4.8× smaller than SBA



6.4 Smooth Instances: Solve Times

Network	Daganzo	Dial	Marcotte–Zhu	SBA	BTA
Braess	(2.38×10^{-1})	0.00678	0.00784	0.00214	$< \mathbf{0.001}$
Sioux Falls	0.473	0.131	0.0597	0.122	**0.032**
Friedrichshain	(2.09×10^{-2})	2.86	0.457	0.259	**0.045**
Berlin Prenzlauerberg	(1.79×10^{-1})	13.7	0.586	2.78	**0.168**
Berlin Tiergarten	(4.41×10^{-1})	2.13	0.0864	0.143	**0.040**
Berlin Mitte	(2.00×10^{-2})	4.01	0.214	0.470	**0.070**
Anaheim	(2.61×10^{-1})	26.9	2.35	1.16	**0.321**
Barcelona	(24.1)	1750	289	38.3	**5.987**
Winnipeg-Asym	(6.22×10^{-2})	611	161	113	**1.275**
Berlin M.-P.-F.	(3.72×10^{-2})	112	1.91	6.47	**1.038**
Winnipeg	(33.5)	2280	823	88.7	**4.673**
Terrassa-Asym	(3.59×10^{-1})	1030	593	496	**9.935**
Hessen-Asym	(3.59×10^{-2})	(1.94×10^{-5})	820	2130	**14.657**
Goldcoast	(112)	(4.71)	(1.37×10^{-2})	$(\mathbf{1.01 \times 10^{-2}})$	(1.09×10^{-2})
Chicago Regional	(3.28×10^{3})	—	(8.67×10^{-1})	(1.85×10^{4})	$(\mathbf{2.03 \times 10^{-1}})$
Berlin Center	(3.44×10^{4})	(1.29×10^{-2})	(1.00)	(2.60×10^{-3})	**25.352**
Philadelphia	(2.21)	—	(1.00)	(24.7)	$(\mathbf{5.06 \times 10^{-4}})$

Solve time $t_{i,m}$ in seconds to reach $\gamma_{i,m}(t) \leq 10^{-5}$; otherwise the final gap $\gamma_{i,m}(T_i)$ in parentheses; "—": no feasible solution saved within T_i. Bold: fastest time, or smallest final gap.

6.5 Nonsmooth Instances

Method	g_m	r_m	h_m	solved
Dafermos	25.641	1.000	0.882	17/17
BTA	204.737	7.985	0.118	15/17

- [Dafermos, 1972] is designed for finitely many classes, that is, for this setting (one class per atom)

Network	Dafermos	BTA
Braess	0.0545	**0.016**
Sioux Falls	**0.031**	0.191
Friedrichshain	**0.060**	15.671
Berlin Prenzlauerberg	**0.315**	39.797
Berlin Tiergarten	**0.130**	7.915
Berlin Mitte	**0.220**	33.597
Anaheim	**0.670**	140.964
Barcelona	**26.931**	2093.585
Winnipeg-Asym	**26.739**	42.728
Berlin M.-P.-F.	**2.500**	275.643
Winnipeg	**21.447**	84.343
Terrassa-Asym	**25.665**	280.418
Hessen-Asym	493.420	**314.519**
Goldcoast	**3216.493**	(5.32×10^{-3})
Chicago Regional	**12385.983**	20352.302
Berlin Center	**420.221**	2014.743
Philadelphia	**36485.144**	(1.65×10^{-2})

Solve time (s) to $\gamma \leq 10^{-3}$, or final gap in parentheses

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- Dafermos is faster on 15 of the 17 instances; BTA is faster on Braess and Hessen-Asym

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- Dafermos is faster on 15 of the 17 instances; BTA is faster on Braess and Hessen-Asym
- BTA reaches 10^{-3} on all instances except Goldcoast and Philadelphia
- BTA solves the restricted problems to the final accuracy, instead of stopping earlier to generate new paths

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Braess	0.0545	**0.016**
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6.6 Tolls and Path Diversity

Fraction of OD pairs with positive equilibrium flow on more than one path:

Network	Instance	No toll	MC $\alpha_0 = 1$	MC $\alpha_0 = 18$	Capacity-UE
Sioux Falls	smooth	0.265	0.394	0.286	0.345
	nonsmooth	0.222	0.570	0.697	0.540
Anaheim	smooth	0.286	0.583	0.762	0.323
	nonsmooth	0.337	0.764	0.925	0.698
Berlin Tiergarten	smooth	0.222	0.484	0.630	0.247
	nonsmooth	0.238	0.618	0.835	0.315
Winnipeg-Asym	smooth	0.788	0.827	0.722	0.789
	nonsmooth	0.786	0.882	0.919	0.829

- No toll is the single-criterion baseline; Capacity-UE tolls the arcs whose single-criterion equilibrium flow exceeds capacity

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	nonsmooth	0.786	0.882	0.919	0.829

- No toll is the single-criterion baseline; Capacity-UE tolls the arcs whose single-criterion equilibrium flow exceeds capacity
- With a second criterion, more OD pairs split their flow over several paths, and in most (but not all) cases the fraction increases with the tolls

Concluding Remarks

- A finite-dimensional convex formulation in the path fractions q for weight distributions with bounded support, including atoms, densities, and gaps

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- Smooth instances: BTA is the fastest method on all 14 instances solved by some method
- Finite support: the method of [Dafermos, 1972] is faster; stopping the restricted problems early to generate paths sooner requires a different algorithm, and is an interesting topic for future research

Some Related Works

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