Discrete Optimization
ISyE 6662 - Spring 2023
Homework 3

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1. Let $G = (N, A)$ be a directed graph, $s, t \in N$ and let $w_a \in \mathbb{Q}$ be arc weights. Recall that a directed path is a sequence of arcs $P = (a_1, \ldots, a_k)$ in which a_l's head is a_{l+1}'s tail, and in which no node repeats. Show that the decision version of directed TSP polynomially reduces to asking if some directed s-t path in G has total weight less than some number.

   Answer: From $G = (N, A)$, we create another directed graph $\widetilde{G} = (\widetilde{N}, \widetilde{A})$ as follows:

   - Duplicate each $n \in N$ and denote by $n', n'' \in \widetilde{N}$.
   - For every arc that arrives to n in G it does arrive to n' in $\widetilde{G}$, and every arc that departs from n in G it does depart from n'' in $\widetilde{G}$. Those arcs have the same weight $w_a \in \mathbb{Q}$.
   - Create an arc from n' to n'' with weight $-M$, where $M = 2\sum_{a \in A} |w_a| + 1$.

   Fixed $n', n'' \in \widetilde{N}$, we prove that the TSP instance has a solution with total weight less than or equal to d if and only if there exists a directed $n' - n''$ path in $\widetilde{G}$ with total weight less than or equal to $d - M(|N| - 1)$, where $|d| \leq \sum_{a \in A} |w_a|$.

   Indeed, given a TSP solution with total weight less than or equal to d, the corresponding sequence of arcs in $\widetilde{G}$ can be completed to a directed $n' - n''$ *Hamiltonian* path by including the transition arcs from a node m' to m'', for every $m', m'' \in \widetilde{N} \backslash \{n', n''\}$. The total weight of such directed path is less than or equal to $d - M(|N| - 1)$ since there will be $|N| - 1$ arcs of weight $-M$ in any directed $n' - n''$ Hamiltonian path.

   Conversely, consider a directed $n' - n''$ path with total weight less than or equal to $d - M(|N| - 1)$. We have to prove that such directed path is *Hamiltonian*, that is, it traverses every pair of nodes (m', m''), where $m', m'' \in \widetilde{N} \backslash \{n', n''\}$. If it does not traverse some pair (m', m'') then $-\sum_{a \in A} |w_a| - M(|N| - 1)$ is a lower bound on the total weight of the directed path. However, this lower bound implies the following inequality

   $$d - M(|N| - 1) \geq -\sum_{a \in A} |w_a| - M(|N| - 2) \quad \iff \quad d + \sum_{a \in A} |w_a| \geq M$$
   $$\implies 2\sum_{a \in A} |w_a| \geq M,$$

   which is a contradiction. Therefore, a directed $n' - n''$ path with total weight $d - M(|N| - 1)$ must traverse all the nodes in $\widetilde{G}$. Thus, the induced sequence of arcs in G defines a *Hamiltonian* cycle with total cost less than or equal to d.

2. Recall the uncapacitated facility location problem. We have a set of candidate locations $M = \{1, \ldots, m\}$, and a set of customers $N = \{1, \ldots, n\}$. Opening a facility at location $i \in M$ incurs a fixed cost of $f_i \geq 0$, and satisfying j's demand from i incurs a cost of $c_{ij} \geq 0$. Customers can only be served from i if the facility

is open, but there is no other constraint (such as capacity) on what the facility can serve. Show that the decision version is NP-complete using 3-SAT.

Answer: The construction of the Uncapacitated Facility Location (UFL) instance from 3-SAT is similar to the Vertex Cover construction. Indeed, we define the same graph from the 3-SAT instance and interpret the nodes as facilities and the edges as customers.

Indeed, consider an instance $(\mathcal{U}, \mathcal{C})$ of the 3-SAT such that $\mathcal{U} := \{x_1, \ldots, x_\nu\}$ is the set of variables and $\mathcal{C} := \{C_1, \ldots, C_k\}$ is the set of clauses. Then, construct a graph $G = (N, E)$ in the following way:

- For each variable $x \in \mathcal{U}$, create a pair of nodes $[x]$ and $[\overline{x}]$. We refer to $[x]$ and $[\overline{x}]$ as *variable nodes.* Connect $[x]$ and $[\overline{x}]$ by an edge called *variable edge.*
- For each clause $C_i \in \mathcal{C}$, create three nodes denoted by $[C_i, l_{i,1}]$, $[C_i, l_{i,2}]$, and $[C_i, l_{i,3}]$, where $l_{i,1}$, $l_{i,2}$, and $l_{i,3}$ are the three literals of C_i. We refer to those nodes as *clause nodes.* Connect all the three clause nodes by an edge and form a clique of size 3. Those edges are called *clause edges.*
- Connect a variable node $[u]$ to a clause node $[C_i, l_{ij}]$ if the literals l_{ij} and u are the same. Call this edge a *forcing edge.*

A few remarks are instructive for this graph.

(i) The number of nodes is $|N| = 2\nu + 3k$ and the number of edges is $|E| = 2\nu + 6k$.

(ii) A lower bound on the size of a vertex cover for G is $\nu + 2k$. This is because the pair of variable nodes $[x]$ and $[\overline{x}]$ form a clique of size 2 and the variable clauses $[C_i, l_{i,1}]$, $[C_i, l_{i,2}]$, and $[C_i, l_{i,3}]$ form a clique of size 3. Recall that to cover the edges of a clique of size r one needs at least $r - 1$ nodes from the clique.

We now define a UFL instance whose solution is essentially the minimum cardinality vertex cover. Indeed, let $f_i = 1$ for every node $i \in N$ and let

$$c_{ie} = \begin{cases} 0, & \text{if } i \in e, \\ M, & \text{if } i \notin e, \end{cases}$$

for all $i \in N$ and $e \in E$, where $M := |N| + 1$. Then, our UFL instance is defined as

$$\begin{array}{lll} \min & \sum_{i \in N} \sum_{e \in E} c_{ie} z_{ie} + \sum_{i \in N} x_i & \\ \text{s.t.} & \sum_{i \in N} z_{ie} = 1, & \forall e \in E, \\ & \sum_{e \in E} z_{ie} \leq |E| \cdot x_i, & \forall i \in N, \\ & x_i \in \{0, 1\}, \ z_{ie} \geq 0, & \forall i \in N, \ \forall e \in E. \end{array}$$

Note that any vertex cover $D \subseteq N$ induces a UFL solution with objective cost equal to $|D|$, and any feasible solution (x, z) to the UFL instance such that the set $\{i \in N \mid x_i = 1\}$ is not a vertex cover has objective cost greater than $|N|$. In particular, $\nu + 2k$ is a lower bound for the optimal value of the UFL instance

We now complete the proof by showing that the 3-SAT instance is *satisfiable* if and only if there exists a feasible solution to the UFL instance with objective cost less than or equal to $\nu + 2k$, or in other words, a vertex cover of cardinality $\nu + 2k$.

Indeed, suppose the 3-SAT is satisfiable and let A be an assignment that makes all the clauses true. Consider the subset of nodes $D \subseteq N$ defined as follows:

- The subset D contains all the variable nodes $[u]$ such that the literal u is true by the assignment A.
- For each clause C_i, the subset D contains the other two clause nodes $\{[C_i, l_{i,j}]\}_{j=1,\ j \neq r}^{3}$ if some literal $l_{i,r}$ is true by the assignment A.

Note that D has cardinality $\nu + 2k$. We prove that D is a vertex cover. Indeed, each variable edge induced by the nodes $[x]$ and $[\overline{x}]$ is covered by exactly one node in D. The clause edges in the clique defined by the node clauses $[C_i, l_{i,1}]$, $[C_i, l_{i,2}]$, and $[C_i, l_{i,3}]$ are covered by exactly 2 node in D. The forcing edge between a variable node $[u]$ and a clause node $[C_i, u]$ is covered by $[u] \in D$ if the literal u is assigned true in A or it is covered by $[C_i, u]$ otherwise. Therefore, D is a vertex cover of cardinality $|D| = \nu + 2k$.

Conversely, suppose that G has a vertex cover $D \subseteq N$ of cardinality $|D| = \nu + 2k$. Then, exactly one variable node among $[x]$ and $[\overline{x}]$ belongs to D, for each variable $x \in \mathcal{U}$, and exactly two node clauses among $[C_i, l_{i,1}]$, $[C_i, l_{i,2}]$, and $[C_i, l_{i,3}]$ belongs to D, for each clause $C_i \in \mathcal{C}$. Thus, the literals of the selected variable nodes can be made true and they induce a Boolean assignment A of the variables in $\mathcal{U}$. For each clause C_i, the one node clause $[C_i, u]$ that does not belong to D is connected to the node variable $[u]$ that must belong to D, otherwise the corresponding forcing edge is not covered by D. This proves that the assignment satisfies all the clauses and the 3-SAT is satisfiable.

3. Consider a knapsack feasible set $S = x \in \{0,1\}^N : \sum_{i \in N} a_i \leq b$, where we assume $a_i \leq b$ for any $i \in N$, i.e. every item can individually fit in the knapsack, and thus S is full-dimensional. Consider a set $C \subseteq N$ satisfying $\sum_{i \in C} a_i > b$.

   (a) Prove that $\sum_{i \in C} x_i \leq |C| - 1$ is valid for S.

   Answer: Because the knapsack coefficients a_i's are non-negative we have the inequality $\sum_{i \in C} a_i x_i \leq b$. If a solution x satisfies $\sum_{i \in C} x_i \geq |C|$ then x_i equals 1 for every node $i \in C$ but this violates the condition $\sum_{i \in C} a_i x_i \leq b$. Thus, the inequality $\sum_{i \in C} x_i \leq |C| - 1$ is valid for S.

   (b) Give necessary and sufficient conditions for the inequality to be facet-defining for $\text{conv}(S)$.

   Answer: We prove that the necessary and sufficient condition are

   (i) The cover C is a *minimal cover*, that is, $\sum_{i \in C \setminus \{j\}} a_i \leq b$, for all $j \in C$.

   (ii) There exists $k \in C$ such that $\sum_{i \in C \setminus \{k\}} a_i + a_n \leq b$, for all $n \in N \setminus C$.

   Indeed, suppose that conditions (i) and (ii) hold. Denote by $\mathbf{1}_C \in \{0,1\}^N$ the vector with 1's at the coordinates $i \in C$, and 0's otherwise. Then, $\{\mathbf{1}_C - e_i : i \in C\} \cup \{\mathbf{1}_C - e_k + e_n : n \in N \setminus C\}$ are $|N|$ affinely independent vectors that belong to the face F induced by the valid inequality $\sum_{i \in C} x_i \leq |C| - 1$, that is,

$$F := \text{conv}(S) \cap \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in C} x_i = |C| - 1 \right\}.$$

   So, F has dimension greater than or equal to $|N| - 1$. The constraint $\sum_{i \in C} x_i = |C| - 1$ is not an implicit equality of $\text{conv}(S)$ since $\text{conv}(S)$ is full dimensional. This implies that the dimension of F is less than or equal to $|N| - 1$. Thus, F has dimension $|N| - 1$ and it is a facet of $\text{conv}(S)$.

   Conversely, suppose that F is a facet of $\text{conv}(S)$. Assume by contradiction that C is not a minimal cover, i.e., there exists a proper subset $C' \subsetneq C$ that is also a cover. This implies that the valid inequality $\sum_{i \in C} x_i \leq |C| - 1$ can be obtained by the summation of the valid inequalities $\sum_{i \in C'} x_i \leq |C'| - 1$ and $x_r \geq 0$ for all $r \in C \setminus C'$. So, the valid inequality $\sum_{i \in C} x_i \leq |C| - 1$ is redundant, so it is not facet-defining for $\text{conv}(S)$, which is a contradiction. Thus, C must be a minimal face and condition (i) is necessary.

   Let $k \in C$ be such that $\sum_{i \in C \setminus \{k\}} a_i$ is minimum. Assume by contradiction that there exists $n \in N \setminus C$ such that $\sum_{i \in C \setminus \{k\}} a_i + a_n > b$. Then, F is contained into the affine space

$$H := \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in C} x_i = |C| - 1, \; x_n = 0 \right\}.$$

   Because H has dimension $|N| - 2$ we conclude that F is not a facet, which is a contradiction. Thus, condition (ii) is also necessary.

4. Let (N, E) be an undirected, connected network, and let $S \subseteq \{0,1\}^E$ be the set of indicator vectors of spanning trees. For each question below, justify your answer with a proof.

(a) Can S ever be full-dimensional?

Answer: No, because S is contained into the proper affine subspace $H := \{x \in \mathbb{R}^E \mid \sum_{e \in E} x_e = |N| - 1\}$.

(b) Suppose the network is itself a tree. What is $\dim(S)$?

Answer: The dimension of S is $\dim(S) = 0$ since the only feasible solution is the tree itself. In other words, the cardinality of S is 1.

(c) Suppose the network is a cycle. What is $\dim(S)$?

Answer: The dimension of S is $\dim(S) = |E| - 1$. Indeed, $\dim(S)$ is less than or equal to $|E| - 1$, and $\{\mathbf{1}_E - e_l \mid l \in E\}$ are $|E|$ affinely independent indicator vectors in S.

(d) What is $\dim(S)$ in the general case? Use your previous answers.

Answer: It was announced in Canvas by professor Toriello.

5. Let (N, A) be a complete directed network, and let $S \subseteq \{0, 1\}^A$ be the set of indicator vectors of directed Hamiltonian cycles. What is $\dim(S)$? Justify your answer.

Answer: The dimension of S is

$$\dim(S) = (|N| - 1)(|N| - 2) - 1.$$

In order to prove this formula, we reduce the problem of a Hamiltonian *cycle* in a complete directed graph with n nodes to the Hamiltonian *path* in a complete directed graph with $n - 1$ nodes.

Indeed, given any enumeration of the nodes $N = \{v_1, \ldots, v_n\}$, a Hamiltonian cycle C in a complete directed graph K_n is the indicator vector of the arcs in the following sequence nodes:

$$C = v_1 v_{\sigma(2)} v_{\sigma(3)} \cdots v_{\sigma(n)} v_1,$$

where $\sigma : \{2, 3, \ldots, n\} \to \{2, 3, \ldots, n\}$ is a permutation, i.e., bijection. Note that $P = v_{\sigma(2)} v_{\sigma(3)} \cdots v_{\sigma(n)}$ defines a Hamiltonian path in the completed directed graph K_{n-1}, where the nodes are given by $N \backslash \{v_1\}$. Thus, there is a one to one correspondence between the set S of indicator vectors of directed Hamiltonian *cycles* in K_n and the set S' of indicator vectors of directed Hamiltonian *paths* in K_{n-1}. In particular, the number of maximal affinely independent vectors are the same. Hence, both set dimensions are the same, i.e., $\dim(S) = \dim(S')$.

So, it is enough to prove that the dimension of the set $\widetilde{S}$ of indicator vectors of directed Hamiltonian paths on a complete directed network $(\widetilde{N}, \widetilde{A})$ is $|\widetilde{N}|(|\widetilde{N}| - 1) - 1$. Indeed, the cardinality of $\widetilde{A}$ is $|\widetilde{N}|(|\widetilde{N}| - 1)$ and for every indicator vector x of a Hamiltonian path we have that $\sum_{a \in \widetilde{A}} x_a = |\widetilde{N}| - 1$. So, the following upper bound holds:

$$\dim(\widetilde{S}) \leq |\widetilde{N}|(|\widetilde{N}| - 1) - 1.$$

Now we show that any implicit equality $\sum_{a \in \widetilde{A}} \alpha_a x_a = \beta$ for $\widetilde{S}$ is a multiple of $\sum_{a \in \widetilde{A}} x_a = |\widetilde{N}| - 1$. Indeed, Given two arcs $a', a'' \in \widetilde{A}$ let H be a directed Hamiltonian cycle containing a' and a''. Then, $H \backslash \{a'\}$ and $H \backslash \{a''\}$ are Hamiltonian paths that we represent by the indicator vectors x' and x'', respectively. Then,

$$\left.\begin{array}{l} \sum_{a \in \widetilde{A}} \alpha_a x'_a = \beta \\ \sum_{a \in \widetilde{A}} \alpha_a x''_a = \beta \end{array}\right\} \implies 0 = \sum_{a \in \widetilde{A}} \alpha_a (x'_a - x''_a) = \alpha_{a'}(0 - 1) + \alpha_{a''}(1 - 0)$$

$$\implies \alpha_{a'} = \alpha_{a''}.$$

Hence, there exists $c \in \mathbb{R}$ such that $\alpha_a = c$ for all $a \in \widetilde{A}$. In particular, we have that

$$\beta = \sum_{a \in \widetilde{A}} \alpha_a x_a = c \cdot \sum_{a \in \widetilde{A}} x_a \quad \Longrightarrow \quad \beta = (|\widetilde{N}| - 1)/c,$$

if c is non-zero. This concludes that any implicit equality $\sum_{a \in \widetilde{A}} \alpha_a x_a = \beta$ for $\widetilde{S}$ is a multiple of $\sum_{a \in \widetilde{A}} x_a = |\widetilde{N}| - 1$. Therefore, $\dim(\widetilde{S}) = |\widetilde{N}|(|\widetilde{N}| - 1) - 1$.

6. Let (N, E) be an undirected network. Recall that a node (or vertex) cover $V \subseteq N$ is a node set that is incident to every edge in E, and let $S \subseteq \{0,1\}^N$ be the set of indicator vectors of covers.

(a) Show that S is full-dimensional.

Answer: The vertex covers defined by $\{\mathbf{1}_N\} \cup \{\mathbf{1}_N - e_i : i \in N\}$ are $|N| + 1$ affinely independent vectors. Thus, S is full-dimensional.

(b) Let $K \subseteq N$ be a clique in the network. Show that $\sum_{i \in K} x_i \geq |K| - 1$ is valid for S. Give necessary and sufficient conditions for the inequality to be facet-defining, and justify these conditions constructively (i.e. by exhibiting n affinely independent points).

Answer: Suppose there is a vertex cover $x \in S$ such that $\sum_{i \in K} x_i \leq |K| - 2$. Then, there are at least 2 nodes $j, k \in K$ that are not in the cover V induced by x. So, the edge $(j, k) \in E(K)$ is not covered by V, which is a contradiction. Thus, the inequality $\sum_{i \in K} x_i \geq |K| - 1$ is valid.

The necessary and sufficient condition for the inequality $\sum_{i \in K} x_i \geq |K| - 1$ to be facet-defining are:

(i) K is a maximal clique, that is, $K \cup \{v\}$ is not a clique, for all $v \in N \backslash K$.

(ii) For all $v \in N \backslash K$, there exists $k = k(v) \in K$ such that $V := N \backslash \{k, v\}$ is a vertex cover.

First, we prove that conditions (i) and (ii) are sufficient. The face F defined as

$$F = \operatorname{conv}(S) \cap \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in K} x_i = |K| - 1 \right\}.$$

has $|N|$ affinely independent vectors given by $\{\mathbf{1}_N - e_j : j \in K\} \cup \{\mathbf{1}_N - e_v - e_{k(v)} : v \in N \backslash K\}$. Thus, F is a facet.

Conversely, suppose that F is a facet. To show condition (i), assume by contradiction that K is not a maximal clique, i.e., there exists $v \in N \backslash K$ such that $K \cup \{v\}$ is a clique. Then, the inequality $\sum_{i \in K} x_i \geq |K| - 1$ is the sum between the valid inequalities $\sum_{i \in K \cup \{v\}} x_i \geq |K|$ and $-x_v \geq -1$. So, the valid inequality $\sum_{i \in K} x_i \geq |K| - 1$ is redundant and cannot be facet-defining, which is a contradiction.

To show condition (ii), assume by contradiction that there exists $v \in N \backslash K$ such that the subset $V := N \backslash \{k, v\}$ is not a vertex cover for all $k \in K$. This implies that the face F is contained in the proper affine subspace $H = \{x \in \mathbb{R}^N \mid \sum_{i \in K} x_i = |K| - 1,\; x_v = 1\}$. So, F is not a facet and this is a contradiction.

(c) Let $C \subseteq N$ be the node set of an odd cycle in the network. Show that $\sum_{i \in C} x_i \geq \lceil |C|/2 \rceil$ is valid for S. Suppose C has a chord, i.e. an additional edge connecting two nodes besides the edges in the cycle. Show that the inequality is not facet-defining. Suppose C is chordless; is the inequality always facet-defining?

Answer: Since $x \in S$ represents a vertex cover, we have that $x_i + x_j \geq 1$ for all $\{i, j\} \in E$. Then,

$$|C| \leq \sum_{\substack{i,j \in C \\ \{i,j\} \in E}} (x_i + x_j) \underbrace{=}_{\text{(cycle)}} 2 \sum_{i \in C} x_i \quad \Longrightarrow \quad \left\lceil \frac{|C|}{2} \right\rceil \leq \sum_{i \in C} x_i.$$

Thus, the inequality $\sum_{i \in C} x_i \geq \lceil |C|/2 \rceil$ is valid for S.

We show that such valid inequality is not facet-defining if C has a chord. Denote by F the face induced by the valid inequality $\sum_{i \in C} x_i \geq \lceil |C|/2 \rceil$. Then, we prove that

(i) There is an odd subcycle C' and an even subcycle C'' formed by the chord in C such that

$$|C| = |C'| + |C''| - 2. \tag{1}$$

Note that equation (1) implies that $\lceil |C|/2 \rceil = \lceil |C'|/2 \rceil + |C''|/2 - 1$.

(ii) The face F is contained into the affine subspace

$$H := \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{l} \sum_{i \in C} x_i = \lceil |C|/2 \rceil, \\ \sum_{i \in C'} x_i = \lceil |C'|/2 \rceil \end{array} \right\}.$$

Thus, F is not a facet.

We first prove (i). Indeed, a chord on a cycle C creates two subcycles C' and C'', where the union $C' \cup C''$ is the cycle C and the intersection $C' \cap C''$ is a set with the two nodes from the chord. If the number of nodes on both subcycles were even or odd then C would be an even cycle. Thus, there must exist an odd subcycle C' and an even subcycle C''.

Now we prove (ii). For that we need to prove the inequality $\sum_{i \in C'' \setminus \{u,v\}} x_i \geq |C''|/2 - 1$ is valid for S, where u and v are the two nodes from the chord in C, i.e., $C' \cap C'' = \{u, v\}$. Indeed,

$$\begin{aligned} |C''| - 3 &\leq \sum_{\substack{(i,j) \in E(C'') \\ i \neq u,v \text{ and } j \neq u,v}} x_i + x_j \\ &\leq 2 \cdot \Bigg(\sum_{i \in C'' \setminus \{u,v\}} x_i \Bigg). \end{aligned}$$

This implies that $\sum_{i \in C'' \setminus \{u,v\}} x_i \geq \lceil (|C''| - 3)/2 \rceil = |C''|/2 - 1$. Since C' is an odd cycle we know the inequality $\sum_{i \in C'} x_i \geq \lceil |C'|/2 \rceil$ is valid for S. Then, for every solution $x \in S$ that belongs to the face F we have that

$$\begin{aligned} \left\lceil \frac{|C|}{2} \right\rceil = \sum_{i \in C} x_i &= \sum_{i \in C'} x_i + \sum_{i \in C'' \setminus \{u,v\}} x_i \\ &\geq \sum_{i \in C'} x_i + \frac{|C''|}{2} - 1. \end{aligned}$$

This implies that $\sum_{i \in C'} x_i \leq \lceil |C|/2 \rceil - |C''|/2 + 1 = \lceil |C'|/2 \rceil$. Hence, $\sum_{i \in C'} x_i = \lceil |C'|/2 \rceil$. Therefore, F is contained into the affine space H.

We now show that even if C is chordless the inequality $\sum_{i \in C} x_i \geq \lceil |C|/2 \rceil$ may not be facet-defining. Let $G = (N, E)$ be the graph defined as $N = \{1, 2, 3, 4\}$ and $E = \{(1,2), (1,3), (2,3), (1,4), (2,4), (3,4)\}$, see Figure 1. Let C be the chordless odd cycle $\{1, 2, 3\}$. Then, for every solution $x \in S$ in the face F it must also belong to the affine space $H := \{x \in \mathbb{R}^N : \sum_{i=1}^{3} x_i = 2,\ x_4 = 1\}$. Thus, F is not a facet.

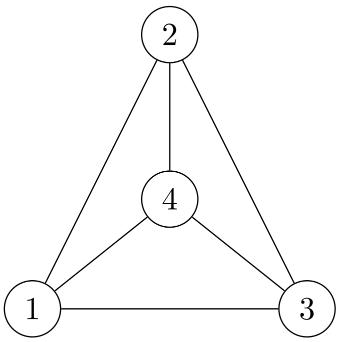

Figure 1: Chordless odd cycle counter-example.