Discrete Optimization
ISyE 6662 - Spring 2023
Homework 6

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1. Let $P_k = \text{conv}\{(0,0),(1,0),(1/2,k/2)\} \subseteq \mathbb{R}^2$ for $k \in \mathbb{Z}_+$. Note that $\text{conv}(P_k \cap \mathbb{Z}_2) = P_0$ for any k. Prove that the CG closure of P_k is P_{k-1} for any $k \in \mathbb{N}$. Hint: You need to show that the inequalities defining P_{k-1} are CG inequalities of P_k, and that the extreme points of P_{k-1} cannot be cut off by any CG inequality of P_k. Your proof shows that the CG rank of P_k is k.

   Answer: Below we have the inequality representation of P_k and the definition of the CG closure of P_k:

$$P_k = \left\{ (x_1,x_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{r} -kx_1 + x_2 \leq 0, \\ kx_1 + x_2 \leq k, \\ x_1, x_2 \geq 0 \end{array} \right\} \quad \text{and} \quad (P_k)' = \bigcap_{\pi \in \mathbb{Z}^2 \setminus \{0\}} \left\{ x \in \mathbb{R}^2 \;\middle|\; \pi^\top x \leq \left\lfloor \max_{\bar{x} \in P_k} \pi^\top \bar{x} \right\rfloor \right\}.$$

   The goal is to prove that $(P_k)' = P_{k-1}$.

   First, we prove that $(P_k)' \subseteq P_{k-1}$. It is enough to show that $-(k-1)x_1 + x_2 \leq 0$ and $(k-1)x_1 + x_2 \leq k-1$ are valid inequalities for $(P_k)'$. Indeed, consider the primal and dual optimization problems:

$$\begin{array}{rlcrll} \max_x & -(k-1)x_1 + x_2 & & = \min_y & ky_2 & \\ \text{s.t.} & -kx_1 + x_2 \leq 0, & (\times y_1) & \text{s.t.} & -ky_1 + ky_2 \geq -(k-1) & (\times x_1) \\ & kx_1 + x_2 \leq k, & (\times y_2) & & y_1 + y_2 \geq 1, & (\times x_2) \\ & x_1, x_2 \geq 0. & & & y_1, y_2 \geq 0. & \end{array}$$

   One can prove using complementary slackness that $(x_1^*, x_2^*) = (1/2, k/2)$ and $(y_1^*, y_2^*) = (1 - 1/2k, 1/2k)$ are primal and dual optimal solutions. Thus, the following is a CG inequality for P_k, that is, a valid inequality for $(P_k)'$:

$$-(k-1)x_1 + x_2 \leq \left\lfloor -(k-1)\frac{1}{2} + \frac{k}{2} \right\rfloor = \left\lfloor \frac{k}{2} \right\rfloor = 0.$$

   Similarly, consider the primal and dual optimization problems:

$$\begin{array}{rlcrll} \max_x & (k-1)x_1 + x_2 & & = \min_y & ky_2 & \\ \text{s.t.} & -kx_1 + x_2 \leq 0, & (\times y_1) & \text{s.t.} & -ky_1 + ky_2 \geq k-1 & (\times x_1) \\ & kx_1 + x_2 \leq k, & (\times y_2) & & y_1 + y_2 \geq 1, & (\times x_2) \\ & x_1, x_2 \geq 0. & & & y_1, y_2 \geq 0. & \end{array}$$

   By complementary slackness, one can prove that $(x_1^*, x_2^*) = (1/2, k/2)$ and $(y_1^*, y_2^*) = (1/2k, 1 - 1/2k)$ are primal and dual optimal solutions. Hence, the following is a CG inequality for P_k:

$$(k-1)x_1 + x_2 \leq \left\lfloor (k-1)\frac{1}{2} + \frac{k}{2} \right\rfloor = \left\lfloor k - \frac{1}{2} \right\rfloor = k-1.$$

Second, we prove that $(P_k)' \supseteq P_{k-1}$. The CG inequality $\pi^\top x \leq \lfloor \max_{\bar{x}\in P_k} \pi^\top \bar{x} \rfloor$ can only cut off fractional extreme points of P_k, which means that we only need to verify whether the extreme point $(1/2,(k-1)/2)$ of P_{k-1} belongs to $(P_k)'$. Since P_{k-1} is a subset of P_k, the valid inequalities of P_k are also valid for P_{k-1}. Hence, the CG cuts $\pi^\top x \leq \lfloor \max_{\bar{x}\in P_k} \pi^\top \bar{x} \rfloor$ that could potentially cut off $(1/2,(k-1)/2)$ are those such that $(x_1^*, x_2^*) = (1/2, k/2)$ is a maximizer of $\max_{\bar{x}\in P_k} \pi^\top \bar{x}$. Others CG cuts for which $(1,0)$ or $(0,0)$ are maximizers of $\max_{\bar{x}\in P_k} \pi^\top \bar{x}$ are valid cuts for P_k.

We now describe the set of integral coefficients $\pi \in \mathbb{Z}^2\backslash\{0\}$ such that $(1/2, k/2)$ is primal optimal. Indeed, consider the primal and dual optimization problems:

$$\begin{array}{rlllrll} \max_x & \pi_1 x_1 + \pi_2 x_2 & & = & \min_y & k y_2 & \\ \text{s.t.} & -k x_1 + x_2 \leq 0, & (\times y_1) & & \text{s.t.} & -k y_1 + k y_2 \geq \pi_1 & (\times x_1) \\ & k x_1 + x_2 \leq k, & (\times y_2) & & & y_1 + y_2 \geq \pi_2, & (\times x_2) \\ & x_1, x_2 \geq 0. & & & & y_1, y_2 \geq 0. & \end{array}$$

If $(x_1^*, x_2^*) = (1/2, k/2)$ is primal optimal then by complementary slackness we have that (π_1, π_2) must have the form:

$$\begin{aligned} \pi_1 &= -k y_1 + k y_2, \\ \pi_2 &= y_1 + y_2, \\ & y_1, y_2 \geq 0, \end{aligned}$$

for some $(y_1, y_2) \in \mathbb{R}^2$. In particular, we have that $\pi_2 \geq 1$ otherwise if $\pi_2 = 0$ then $\pi_1 = 0$ as well. Thus, the left-hand side of the cut $\pi^\top x \leq \lfloor \max_{\bar{x}\in P_k} \pi^\top \bar{x} \rfloor$ evaluated at $(1/2,(k-1)/2)$ is of the form

$$\begin{aligned} \frac{\pi_1}{2} + \frac{k-1}{2}\pi_2 &= \frac{-k y_1 + k y_2}{2} + \frac{(k-1)y_1 + (k-1)y_2}{2} = \frac{-(y_1 + y_2) + 2k y_2}{2} \\ &\leq -\frac{1}{2} + k y_2 \leq \lfloor k y_2 \rfloor. \end{aligned}$$

The last inequality $-1/2 + k y_2 \leq \lfloor k y_2 \rfloor$ holds because $k y_2 = (\pi_1 + k\pi_2)/2$, so if $k y_2$ is not integer then its fractional part is $1/2$. The right-hand side of the cut $\pi^\top x \leq \lfloor \max_{\bar{x}\in P_k} \pi^\top \bar{x} \rfloor$ is

$$\left\lfloor \frac{\pi_1}{2} + \frac{k}{2}\pi_2 \right\rfloor = \left\lfloor \frac{-k y_1 + k y_2}{2} + \frac{k y_1 + k y_2}{2} \right\rfloor = \lfloor k y_2 \rfloor .$$

Hence, $(P_k)' \supseteq P_{k-1}$.

2. For n odd, consider

$$\min x_0 \qquad \text{s.t. } x_0 + 2\sum_{i=1}^{n} x_i = n, \qquad x \in \{0,1\}^{n+1}. \tag{1}$$

a) Prove that a branch-and-bound algorithm that branches on individual variables needs to evaluate exponentially many nodes to solve this problem.

Answer: Consider the linear relaxation of the original problem (1) with right-hand side n replaced by k and l variables:

$$\begin{array}{rll} \nu(k,l) = \min & x_0 & \\ \text{s.t.} & x_0 + 2\sum_{j=1}^{l} x_j = k, & \\ & 0 \leq x_j \leq 1, & j = 1, \ldots, l. \end{array} \tag{2}$$

Suppose that k is an odd integer. Then, we have a few possibilities:

- If $2l + 1 > k$ then $\nu(k,l) = 0$ and every optimal solution $x^* \in [0,1]^l$ to (2) must have a fractional coordinate.
- If $2l + 1 = k$ then the only feasible solution is $x_j = 1$ for every j. In particular, $\nu(k,l) = 1$.
- If $2l + 1 < k$ then (2) is infeasible and $\nu(k,l) = +\infty$.

In summary, we have that

$$\nu(k,l) = \begin{cases} 0, & \text{if } 2l+1 > k, \\ 1, & \text{if } 2l+1 = k, \\ +\infty, & \text{if } 2l+1 < k. \end{cases}$$

The best possible upper bound for the original integral problem (1) is 1, which is the optimal value. The branch-and-bound algorithm that branches on individual variables will

(1) Initialize a queue of nodes with the root node, that is, the linear relaxation $\nu(n,n)$ of the original problem (1).

(2) Pop a node $\nu(k,l)$ from the queue and solve the corresponding linear relaxation problem.

(3)
- If $\nu(k,l)$ equals the upper bound 1 then the algorithm stops with a certificate of optimality.
- If $\nu(k,l) = +\infty$ then go to step (2).
- Otherwise, the optimal solution x^* of the linear relaxation problem $\nu(k,l)$ is fractional and the branch-and-bound add the node subproblems induced by the additional equality constraints $x_j = 0$ and $x_j = 1$, where j is the fractional optimal solution coordinate. By relabeling the variables, this is equivalent to append the subproblems like (2) with optimal values $\nu(k,l-1)$ and $\nu(k-2,l-1)$ to the queue. Go to step (2).

The data structure of the branch-and-bound subproblems is clearly a binary tree. Moreover, if the traversal of the binary tree is the Depth-First search, that is, we solve *all* the subproblems of each level of the binary tree before we move to the next level, then one requires to visit all the nodes of the first $(n+1)/2$ levels to find the optimal solution. This is because only after the number of variables is reduced to $(n-1)/2$ that the algorithm stops. Hence, it requires to evaluate at least $2^{(n+1)/2-1}$ nodes to solve this problem.

b) Give a CG cut that allows the algorithm to solve the problem at the root node.

Answer: The equality constraint $x_0 + 2\sum_{j=1}^n x_j = n$ and the non-negativity constraint on x_0 implies the valid constraint $\sum_{j=1}^n x_j \le n/2$. Since $n/2$ is fractional we get the CG cut

$$\sum_{j=1}^{n} x_j \le (n-1)/2. \tag{3}$$

The linear relaxation of the original problem (1) with the CG-cut (3) has optimal value 1 and integral optimal solution. Indeed, consider the primal and dual linear programs:

$$\begin{array}{lll} \min_x & x_0 & \\ \text{s.t.} & x_0 + 2\sum_{i=1}^n x_i = n, & \times(w) \\ & \sum_{i=1}^n x_i \le \frac{n-1}{2} & \times(z) \\ & x_j \le 1, & \times(y_j), \\ & & j = 1,\ldots,n. \end{array} \quad \text{and} \quad \begin{array}{lll} \max_{w,z,y} & nw + \frac{n-1}{2}z + \sum_{j=0}^n y_j & \\ \text{s.t.} & w + y_0 \le 1, & \times(x_0) \\ & 2w + z + y_j \le 0, & \times(x_j) \\ & & j = 1,\ldots,n, \\ & w \in \mathbb{R},\ z \le 0,\ y_j \le 0, & j = 1,\ldots,n. \end{array}$$

Let $r = (n-1)/2$ and note that the following are primal x^* and dual feasible solutions (w^*, z^*, y^*) with objective values equal to 1:

$$x_j^* = \begin{cases} 1, & \text{if } 0 \le j \le r, \\ 0, & \text{if } j \ge r+1. \end{cases}, \qquad w^* = 0, \qquad z^* = 0, \qquad y_j^* = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{if } j \ge 1. \end{cases}$$

Hence, the CG cut (3) solves the problem at the root node.

3. Consider a knapsack set $S = \left\{ x \in \{0,1\}^n : \sum_{i=1}^n a_i x_i \le b \right\}$. Suppose the first $k < n$ items form a minimal cover, and relabel them so $a_1 \le \cdots \le a_k$. This implies $\sum_{i=1}^k x_i \le k-1$ is a valid cover inequality for $\text{conv}(S)$.

a) Define the function

$$\tilde{\gamma}(z) = \max \sum_{i=1}^{k} x_i \qquad \text{s.t.} \sum_{i=1}^{k} a_i x_i \le b - z, \qquad \sum_{i=1}^{k} x_i \le k-1, \qquad x \in [0,1]^k.$$

Define $\tilde{\alpha}_i = k - 1 - \tilde{\gamma}(a_i)$ for $i > k$. Prove that the inequality

$$\sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} \tilde{\alpha}_i x_i \le k-1$$

is valid for $\operatorname{conv}(S)$. Note that the coefficients in the inequality do not depend on the order of the items $k+1, \ldots, n$.

Answer: First, the value function $\tilde{\gamma}$ is concave. By Strong Duality, the value function $\tilde{\gamma}$ can be represented as the minimum of affine functions on z, and the minimum of concave functions is concave.

Second, the function defined as $\tilde{\alpha}(z) = k - 1 - \tilde{\gamma}(z)$ is convex and superadditive on $\mathbb{R}_+$, where the latter property is $\tilde{\alpha}(z_1) + \tilde{\alpha}(z_2) \le \tilde{\alpha}(z_1 + z_2)$, for every $z_1, z_2 \ge 0$. Since $\tilde{\gamma}$ is concave we have that $-\tilde{\gamma}$ is convex, so $\tilde{\alpha}$ is a convex function. Also, given $z_1, z_2 \in \mathbb{R}_+$, we have that

$$\begin{aligned}
\tilde{\alpha}(z_1) + \tilde{\alpha}(z_2) &= \tilde{\alpha}\left((z_1+z_2)\frac{z_1}{z_1+z_2}\right) + \tilde{\alpha}\left((z_1+z_2)\frac{z_2}{z_1+z_2}\right) \\
&\le \frac{z_1}{z_1+z_2}\tilde{\alpha}\left(z_1+z_2\right) + \frac{z_2}{z_1+z_2}\tilde{\alpha}\left(z_1+z_2\right) \\
&= \tilde{\alpha}\left(z_1+z_2\right).
\end{aligned}$$

We prove that the inequality described in the statement of this question is valid for $\operatorname{conv}(S)$. Let $x \in S$ and let $I(x) := \{i \mid x_i = 1,\ i \ge k+1\}$. Then,

$$\begin{aligned}
\sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} \tilde{\alpha}_i x_i &= \sum_{i=1}^{k} x_i + \sum_{i \in I(x)} \tilde{\alpha}(a_i) \\
&\le \sum_{i=1}^{k} x_i + \tilde{\alpha}\left(\sum_{i \in I(x)} a_i\right) \\
&= \underbrace{\sum_{i=1}^{k} x_i - \tilde{\gamma}\left(\sum_{i \in I(x)} a_i\right)}_{\le 0} + k - 1 \\
&\le k-1,
\end{aligned}$$

where the last inequality follows from the definition of the value function $\tilde{\gamma}$ since $(x_1, \ldots, x_k)$ is a feasible solution to the corresponding linear program with $z = \sum_{i \in I(x)} a_i$.

b) Prove that the inequality

$$\sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} \lfloor a_i / a_k \rfloor x_i \le k-1$$

is a CG inequality of $\operatorname{conv}(S)$. Hint: You can use $\tilde{\gamma}$.

Answer: Let $x \in S$. In particular, we have that $\sum_{i=1}^{n} a_i x_i \le b$ and $x_i \le 1$, for all $1 \le i \le n$. Because the first coefficients k are ordered $a_1 \le \cdots \le a_k$, we obtain the inequality $\sum_{i=1}^{k-1}(a_k - a_i)x_i \le \sum_{i=1}^{k-1} a_k - a_i$. By summing the latter with the knapsack inequality $\sum_{i=1}^{n} a_i x_i \le b$, we have the following relations:

$$\sum_{i=1}^{k} a_k x_i + \sum_{i=k+1}^{n} a_i x_i \le b + \sum_{i=1}^{k-1} a_k - a_i \qquad \Longleftrightarrow \qquad \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} \frac{a_i}{a_k} x_i \le \frac{b - \sum_{i=1}^{k-1} a_i}{a_k} + k - 1.$$

By taking the floor function of a_i/a_k we get a valid inequality for $\text{conv}(S)$,

$$\sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} \left\lfloor \frac{a_i}{a_k} \right\rfloor x_i \leq \frac{b - \sum_{i=1}^{k-1} a_i}{a_k} + k - 1,$$

and by rounding down the right-hand side we obtain a CG inequality for $\text{conv}(S)$

$$\sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} \left\lfloor \frac{a_i}{a_k} \right\rfloor x_i \leq \left\lfloor \underbrace{\frac{b - \sum_{i=1}^{k-1} a_i}{a_k}}_{\in[0,1)} + k - 1 \right\rfloor = k - 1.$$

The fact that $(b - \sum_{i=1}^{k-1} a_i)/a_k$ belongs to $[0, 1)$ follows from the minimal cover assumption, that is, $\sum_{i=1}^{k-1} a_i \leq b$ and $\sum_{i=1}^{k} a_i \geq b + 1$.